THE

# MATHEMATICAL GAZETTE

EDITED BY
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## THE MATHEMATICAL ASSOCIATION.

A GENERAL Meeting of the Mathematical Association was held at King's College, London, on 5th and 6th April, 1945.

On Thursday, 5th April, the business meeting was held at 10.30 a.m., the President, Mr. C. O. Tuckey, in the chair. The Joint Report of the Council and Executive Committee* for the year 1944 was adopted. On the nomination of the Council, Professor S. Chapman, F.R.S., was elected President for the year 1945. The existing Vice-Presidents, the Treasurer, Secretaries, Librarian, Editor of the *Gazette*, and other present members of the Council were re-elected; the Auditor was re-elected.

The following new rules, dealing with the position of the Association in regard to Income Tax, were proposed by the Treasurer, seconded by Mr. G. L. Parsons, and carried unanimously:

RULE 32 (iv):

The income and property of the Association, whencesoever derived, shall be applied solely towards the promotion of the objects of the Association as set forth in these Rules, and no portion thereof shall be paid or transferred directly or indirectly, by way of dividend, bonus or otherwise howsoever by way of profit, to the members of the Association.

Provided that nothing herein shall prevent the payment, in good faith, of reasonable and proper remuneration to any officer or servant of the Association, or to any member of the Association in return for any service actually rendered to the Association.

RULE 33 (iii):

If in the event of the winding-up or dissolution of the Association there remain, after the satisfaction of all its debts and liabilities, any property whatsoever, the same shall not be paid to or distributed among the members of the Association, but shall be given or transferred to some other institution or institutions having objects similar to the objects of the Association, and which shall prohibit the distribution of its or their income and property among its or their members to an extent at least as great as is imposed on the Association under or by value of Rule 32 (iv) hereof, such institution or institutions to be determined by the Members of the Association at or before the time of dissolution, or in default thereof by a Judge of the High Court of Justice having jurisdiction in regard to charitable funds, and if and so far as effect cannot be given to such provision, then to some charitable object.

At 11 a.m., the President delivered his Presidential address, " Teachers and Examiners ".†

* See pp. 46–68. † See pp. 49–56.

At 2 p.m., a discussion on "Technical Mathematics" was opened by Dr. N. W. McLachlan and Mr. H. V. Lowry; at 5 p.m., Professor P. J. Daniell read a paper on "Integrals in infinitely many dimensions".

On Friday, 6th April, at 10 a.m., Mr. R. A. Fairthorne read a paper on "Some mathematical aspects of punched card accounting machinery and methods", followed at 11 a.m. by a paper and demonstration by Mr. A. P. Rollett on "Mathematical models and constructions". At 2 p.m., a discussion on "Syllabuses for examinations taken by Sixth Form pupils" was opened by Mr. K. S. Snell, Dr. E. A. Maxwell and Mr. J. L. Brereton. At 5 p.m., Dr. J. W. Jenkins spoke on "Statistics and the school course".

A Publisher's Exhibition was open during the two days.

## JOINT REPORT OF THE COUNCIL AND EXECUTIVE COMMITTEE, 1944.

*Membership.*—During the year 1944, 154 new members have been admitted, of whom 20 are junior members. The number of members now on the roll is 1,652, of whom 7 are honorary members, 122 life members, 1,463 are ordinary members and 60 are junior members.

The Council regrets to report the death of the following members of the Association: Miss J. O. Archibald, Mr. C. L. Beaven, Prof. W. E. H. Berwick, Sir A. S. Eddington (President 1930), Miss M. E. Gibson, Sir Joseph Larmor (President 1895–6), Mr. R. M. Milne, Dr. J. McWhan, Sir T. P. Nunn (President 1917–1918), Mr. A. G. Phillips, Prof. C. H. Rowe, Mr. E. H. Slack, Dr. F. J. W. Whipple, Prof. J. R. Wilton, Mr. L. W. Wood, Dr. F. A. Yeldham.

Obituary notices on Sir A. S. Eddington and Sir T. P. Nunn have appeared in recent *Gazettes*. Sir Joseph Larmor was the last surviving Ex-President of the Association for the Improvement of Geometrical Teaching, which became the Mathematical Association in 1897. Prof. W. E. H. Berwick was Chairman of the North Wales Branch of the Association. Prof. J. R. Wilton was well known to Australian members, especially in Adelaide.

The Council offers its congratulations to two Vice-Presidents, whose names appeared in the New Year's Honours List, Prof. E. T. Whittaker being honoured with a Knighthood and Prof. A. N. Whitehead receiving the Order of Merit.

The Council is pleased to report that contact has been re-established with members in France.

*Branches.*—Information with regard to Branch activities is still very incomplete. While some Branches have had a programme somewhat comparable with their pre-war activities, others have so far been unable to resume their meetings. The Mathematical Society of the University of Birmingham has been affiliated as a Junior Branch. The London Branch has held a discussion on the School Certificate syllabuses advocated in the Report of a Conference of Examining Bodies and Teachers' Associations. The Sheffield and District Branch has also discussed syllabus questions and, in spite of exceptional difficulties, the Plymouth Branch has held three meetings. The Yorkshire, North Eastern, Manchester and Midland Branches are also meeting regularly. The Yorkshire Branch has decided to print and circulate to its members the report of its own sub-committee on the question of syllabuses in Secondary and Technical Schools.

The Australian Branches have made steady progress, in spite of the absence of younger members on service. Here, as at home, considerable interest is being shown in the contents of the mathematical syllabus. The Council welcomes the news that a mathematical journal is to be published in Australia

and looks forward with keen interest to the first issue. The Council, while offering its thanks to Branch officials for good work already done, urges those Branches which have not yet resumed activity to consider whether it may not be possible to do so in the near future. In the many educational issues to be discussed, the Association can only play its full part if the views of its members are regularly exchanged and made known. The Council would also like to call the attention of Branch officials to the necessity of rendering annually a report of their activities, as required under Rule 15.

*The Mathematical Gazette.*—The *Gazette* for 1944 makes a volume of 228 pages, the same number as in the preceding volume. This size has only been maintained by a drastic sacrifice of the number of copies available for sale to non-members and for reserve stock. But it has seemed preferable to make this sacrifice in order to prevent a further reduction in the size.

The general war-time policy of giving preference to articles specifically dealing with problems of school mathematics has been continued. Particular reference may be made to Dr. L. J. Comrie's article, "Careers for girls", written in reply to the query printed in the *Gazette* for July, 1943.

The British Council is arranging to send microfilm copies of the *Gazette* to China in response to an urgent request for such copies.

*The Library.*—The Library has been used steadily throughout the year but there are no developments to report.

*The Teaching Committee.*—The activity of the Teaching Committee has again been restricted by war-time conditions. The Trigonometry Sub-Committee has reopened its activities and it is hoped to publish some part of a report in the future. The List of books suitable for School Libraries has been reprinted but it is hoped that work may be started on a revision of this useful report very shortly.

*The Problem Bureau.*—Assistance has been given in the solution of about 120 problems during the period covered by this report. These problems have principally been derived from the Cambridge Scholarship papers and from the various Higher School Certificate Examinations. Many of the members applying were doing so for the first time and some members who had recently joined were included. After a lull during the autumn, a period of renewed activity commenced in the months of January and February.

A number of members have given willing and skilful assistance in providing the necessary solutions, and mention might be made of the assistance given by Mr. R. H. Cobb in dealing with problems in Mechanics.

*General.*—The Advisory Committee for Higher School and Scholarship Mathematics, to which the Association appoints four delegates, has issued an important document setting forth suggested syllabuses for Mathematics at the Higher School Certificate stage. Copies of this are being circulated to members. Mr. R. C. Lyness retires from this Committee and is succeeded by Miss W. M. Lehfeldt. Mr. A. Robson has also retired, his place being taken by Mr. C. O. Tuckey.

Mrs. E. M. Williams and Miss W. M. Lehfeldt represented the Association at an important and useful conference at Cambridge in July on the Scholarship Examinations of Girton and Newnham Colleges in Mathematics, Natural Sciences and Geography.

*Officials and Council.*—The Council records its appreciation of Mr. Tuckey's services as President and nominates Prof. S. Chapman, F.R.S., as President for the year 1945–6.

As meetings of the Council have been much restricted, the Executive Committee recommends that the rule with regard to retirement from the Council be waived and that the existing Council be re-elected.

The Council extends its thanks to Prof. Broadbent for his services as Editor. In the opinion of the Council, the maintenance of the *Mathematical Gazette* at its present high level in spite of war-time restrictions and difficulties has been the principal factor in holding the Association together. The thanks of the Council are also due to Prof. Neville for his work in connection with the Library, another task rendered much more difficult by war-time conditions, and to Mr. Gossett-Tanner and the members who assist him in the Problem Bureau. The services of the Treasurer, the Secretaries are also noted, and, last but not least, those of the Clerical Assistants at Gordon Square who have done a great deal to keep the Association's affairs running smoothly.

## THE ANNUAL GENERAL MEETING: 1946

THE attention of members is called to the postcard circulated with this issue of the *Gazette*. It is of the greatest value to those who are concerned with the planning of the Annual Meeting to have early indication of the views of members. This is especially necessary this year, when the Council has to decide whether to revert to the pre-war custom of holding the Annual Meeting in January (as laid down in the Rules) or whether to continue the recent interim arrangement of a meeting in April. A majority of those present in April this year were, not unexpectedly, in favour of April 1946, but it is possible that many members who could not attend in April might be able to do so in January.

At the next Annual Meeting, the regular procedure for the election of Officers and the Council will be followed. The attention of members is therefore drawn to Rule 21: nominations under this rule should be sent to one of the Secretaries not later than the 21st of October.

G. L. PARSONS } *Honorary*
E. M. WILLIAMS } *Secretaries*

## BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

*Applicants are requested to return all solutions to the Secretary.*

## GLEANINGS FAR AND NEAR.

**1452.** NEWTON A NAZI?

Newton's own leaning seems to have been towards Unitarianism, and the Aryan heresy was anathema to his editor, Bishop Horsley.—H. C. Plummer, *The Observatory*, December 1942, p. 355.

# TEACHERS AND EXAMINERS.

By C. O. Tuckey.

Presidential Address to the Mathematical Association, April, 1945.

I have ventured on my somewhat thorny subject feeling that, having been a teacher in schools for 127 terms, I ought to be able to tell the examiner what the teacher wants from him, and that having been an examiner for let us say ten years more than the average age of the feminine part of my audience, I ought to be able to tell the teacher what the examiner wants from him. (I say "him" but of course "her" is intended equally throughout.)

Dividing examiners into markers of scripts and setters of papers, I shall begin with the former and larger class and ask "What does the teacher want from the marker of examination papers?"

The first demands on the marker are for the virtues of honesty and diligence. He should be completely impartial not only as between candidates but also as between different methods of doing the same question. He must not mark his own pet method up and my pet method down. He must not penalise the answers to other questions because one question is answered in some way that annoys him. He must also be careful and accurate. My pupils must not be ploughed because their marks are added incorrectly or because some page is left unmarked. The candidates (especially my pupils) must also be guarded against their own folly. If they have left several pages blank—in spite of being warned not to do so—and then add further attempts, the examiner must not hastily suppose that they have finished and fail to mark the addenda. Work done on the wrong pages should be marked, and please Mr. Examiner will you give credit for work done, hastily scratched out and not replaced. Also in spite of warnings about "neatness and clearness of expression" you must be prepared to struggle to decipher the untidy scrawls which long practice in my own class-room has at last made *almost* legible to me.

To all these requests Mr. Examiner is not sorry to agree, in spite of the fact that he is always working against time, except for the last one. He would dearly like to be allowed to penalise bad writing and, as you and I do now and then in the class-room, scrawl "illeg." across a page here and there and treat that page as absent. Unfortunately he is not allowed to do this.

What does the examiner—the marker of scripts—ask from the teacher? He might perhaps say to the teacher:

Do you think you could manage to make it clear that what examiners want is five questions well done rather than seven scrawled and hopelessly inaccurate. (It is obvious why we examiners want this: it gives us far less trouble to look over and mark.)

Do please teach your pupils to write legibly and arrange their work properly, so that if they *can* do a question easily, it is done so that it is easy to give it full marks.

As to arrangement of work, let it be plugged in your teaching that examiners don't like arguments ending with " $\therefore\ 2=2$ "; that they take an instinctive dislike to candidates who think that a number is equal to its own logarithm or an angle equal to its own sine; and that they feel that being ploughed is a fate far too merciful for those who send in final answers without any visible work to show how the answers are obtained, except perhaps some on that side of the answer book which is to be used for rough work only and then probably three pages further on than the page opposite the answer. In geometry, if there is a construction, we should like to know whether $A$ has been joined to $C$ and $C$ to

$B$, or whether $A$ has been joined to $B$ and $C$'s presence on the line is guess-work. It is also quite a pleasure to be told what the candidate is trying to prove ; but perhaps that is one of the pleasures of the past.

Train your pupils to do the easy questions first and make sure of them. Legible writing and the easy questions done right is what examiners like. Persuade your pupils that having got three-quarters of the way through a question and having stuck, they should *not* scratch it all out in disgust. If they leave it alone, the partial marks secured for it may make all the difference. " Never delete till you replace " should be the motto.

If a teacher says that he has observed all this from his youth, let him, as a work of supererogation, teach his pupils how to spell " isosceles ".

It is, I think, very important that the teacher and the marker of his pupils' scripts should understand each other's point of view. For this reason it is highly desirable that those who mark school examinations should themselves be or have been teachers in schools.

This is not to say that teachers in colleges are not often highly efficient as markers of quite elementary work, but it is less natural for them. Of course when it comes to Higher Certificate and Scholarship work that is a different matter and *I am not to be taken as saying that College Dons should not mark elementary school work*. But what is really desirable is that all teachers in schools should, at some early period of their careers, take part in marking some large public examinations, up to a minimum total of about a thousand scripts of school certificate standard, implying probably three separate examinations.

If, as it should do, such experience of examination marking produced automatically a substantial rise in the salary which his school paid the teacher, there would be no dearth of assistant examiners.

Having advised all teachers to become examiners I shall here digress from my main theme to ask " What qualities must the teacher possess to enjoy (or not seriously to hate) marking examination papers in bulk? "

He will have to sit long hours at his desk, usually in addition to his ordinary work. 360 scripts at not more than 12 scripts per hour at fastest takes at least 30 hours and " at fastest " is by no means the same as " all through ": the first 50 are *much* slower. How can he enjoy his job?

The examiner needs, I think, a certain childish capacity to be interested and amused by small things. Let him notice how many do Qn. 3 his way and how many do it the other way; which questions are the most popular and which turns out to be the easiest ; the various ways in which one school differs from another ; the various artifices used to conceal ignorance ; how many boys will boldly tell him some question is wrong ; and so on. Such variations of phrase as " the pitch theorem "—for what? for our old friend Pythagoras—should amuse and not annoy the by-nature examiner. He should feel a childish pleasure when analysing the marks in crossing the four strokes with a fifth and should enjoy making small bets with himself as to which marks will be the most and which the least popular. Incidentally the analysing of a large set of marks, say 800 or so, seems to me to provide one of the most suitable opportunities for a sweepstake that I know. The way in which marks race for head place or lag behind is most surprising. Perhaps the perfect examiner, if he studied the paper and the mark scheme, could tell me why in a paper I dealt with last year, 20 candidates got 32 and only 7 got 33, or why 20 candidates got 67 and only 8 got 68, or why the numbers for the marks from 56 to 60 were 15, 14, 24, 16, 15. What had 58 got that made it so easy of attainment?

I am tempted to add another example of what I may call the minor diversions of an examiner. Very recently it has been my lot to mark a large

number of scripts done by elementary school pupils trying for places in secondary schools—not, you may think, a job of much mathematical interest. But there are always these questions of probability.

In the Arithmetic paper the first question consisted of nine easy sums, suitable for mental work, of which the first six were marked 3 or 0 and the other three, which were a little harder, 4 or 0. There was thus a possible total of 30, six 3's and three 4's. Now keeping to the better pupils, I should like you to think for a minute which of the total marks for this question, 24, 25, 26, 27 would be the most likely and why? The answer is not difficult. 26 is a likely mark as it means success in all the sums except one of the harder ones ; 27 is less likely, as it means success in the three harder sums, but failure in one of the easier ones ; 24 is still less likely as it means success in the three harder sums but failure in two of the easier ones ; while 25 is with this mark scheme impossible.

Is it obvious to you which marks between 0 and 30 are impossible? And that if $x$ is impossible, so is $30 - x$? If not, there is a certain—shall I say—pleasant titillation of the mind awaiting you as you realise the facts about the various possibilities while (rather mechanically) recording the results " in pencil of any colour other than red ".

It must not be supposed that it is always easy to know how to mark (even if you can read the candidates' writing). Here is a little point that I had to decide last year in a paper in which I marked all the scripts. It was the first question in a geometry paper and said " Draw a triangle whose sides are 4 cm., 5 cm., 7 cm. Draw all the altitudes, *one of them using ruler and compass only.*" This last phrase, of course, wanted the knowledge of how to construct a perpendicular to a line from an external point, and five marks were set aside for this bit of knowledge. But it so happened that the candidates had *ruled* paper in their answer books and when, as usually occurred, the 7 cm. was drawn along one ruled line, the vertex came, as near as no matter, on another ruled line. Consequently many candidates used the parallel to the base through or nearly through the vertex and gave the ruler and compass construction for a perpendicular to a line from a point *on* it, instead of from a point *outside* it. Now there were three alternatives for the marks : full marks, partial marks, no marks. I will ask for a *quick* show of hands. Those who favour full marks ; partial marks ; no marks. I see that partial marks has it. It would be giving away a secret of the examination to say what the examiner did about it.

As I have advised you all to become examiners I will before I pass on offer one piece of advice to new examiners. *You must not let your eyes get tired.* If you do, your speed drops from say 12 scripts per hour to perhaps 4 per hour. I have found it pay never to work more than 50 minutes in an hour and to take the 10 minutes off, if possible out of doors. This of course adds to the pleasure of life and *it also increases the speed.*

Now I will turn to the examiner as a setter of papers—the chief examiners who of course are also markers.

Here the teacher will say :

One important thing is that you should interpret the syllabus in the same way that I do, stressing those parts that I like to stress and being very careful to set nothing but the easiest questions on those parts that I like to shirk. Surely you should know as well as I do what are the important and interesting parts of mathematics?

The weak spot about this very natural and reasonable request from the teacher, for what I might call local emphasis within the syllabus, is completely spoilt by the difficulty in getting any two teachers to agree as to the parts of the curriculum on which the local emphasis should fall. We all know how diffi-

cult this is for even the teachers in our own particular school: each, it may be said with some confidence, likes to strike out his own line and it is no easy job to persuade him that team-work is the important thing. How much more is this the case with teachers in different schools? It is here that conferences between mathematical teachers in different schools, especially perhaps meetings of Branches of the Mathematical Association, do so much good when they discuss quite elementary subjects.

Anyhow, the net result of the disagreements of which I have spoken is that in this matter of local emphasis and lack of emphasis, teachers are inclined to keep their grievances to themselves—except in the matter of surds.

When the teacher really is vocal, however, is if the examiner sets what is too hard or is outside the syllabus or is slightly unfamiliar in phrasing. It is not too much to say that the teachers singly and in bodies bring steady and constant pressure on the examiners to make their papers *easier*. And the worst of it is that all this constant agitation, these protests leading to head-scratching among examining boards and examiners and moderators, and in fact all concerned, derives all its driving force, all the keenness and vigilance given to it, from a complete misunderstanding of the position. The teacher says to himself "This paper is too hard for my precious rabbit" (technical term for duffer). "If only the examiner would make things easier, my rabbit could understand the questions and do them well enough to get that pass or that credit which he wants so badly. So I shall go for that examiner. I shall point out that the second part of Qn. 10 is suitable only for those with brains and that the second part of Qn. 1 is worded so that my rabbit does not know what he is allowed to assume in proving it; that there are two ways of reading Qn. 7 and that though one of these makes complete nonsense, my rabbit cannot be expected to realise that."

However, there is at least one teacher who takes the other view. I had the odd experience not long ago that a teacher came up to me and said, "The certificate papers which you and your colleagues set were delightfully straightforward and easy. There seemed to be nothing which worried or frightened even my timidest rabbit and they all of them came away from the examination feeling that they really had had a chance and had taken it. Unfortunately the result was that all my doubtfuls failed to get their credits."

I have said that this was an unusual and odd experience and to be strictly truthful it only came to me in a dream. But to one who knows how examinations are run, there is nothing really odd about what was supposed to happen, and my dream function is one with no "essential singularity". Things might easily happen so. If the papers are "delightfully straightforward and easy" for Mr. X's rabbits, they will be so also for Mr. Y's and Miss Z's, and these latter may write a little faster or be just a bit more accurate than Mr. X's, and by comparison Mr. X's pupils come just a bit lower in the final order of merit among the candidates than they would have done if the papers had been more full of guile. The point that cannot be too clearly understood by teachers is that to get credit in the School Certificate it is *not* necessary for the pupil to get 50% of the marks, but it *is* necessary for him to be in the first 50% of the candidates. I take 50% as a round number for convenience. Of the two 50's one should perhaps be 60 and the other 54.

If it has not the effect of ploughing more candidates what is the effect of making the examination harder? The answer, I think, is that various ways of leaving the perfectly straightforward commonplace type of question which is well within the syllabus gives a better chance to one type of candidate and a worse chance to another type. The perfectly straightforward commonplace type favours the quick worker even if he is a bit inaccurate, and penalises the slow, accurate worker. The paper which is inclined to go beyond the syllabus

favours the pupils of those masters who make a habit of going beyond the syllabus—I was always one of them—and penalises those who are very carefully crammed for the examination and the examination only. The paper which starts with a hard question hits the nervous pupil who suddenly feels "I can't do even No. 1; it is hopeless" and gets really flurried. Incidentally, teachers might ask themselves "Is the type that is flurried by an initial failure and packs up, the type that should secure responsible and important posts by success in examinations?" The paper which is worded in a slightly unusual manner penalises the candidate who cannot attach a meaning to a bit of English but only knows what a question is about if it is one of a type that he done over and over again. And so on and so forth.

The set of papers which is definitely too hard, containing in the ordinary papers questions more suitable for Advanced Mathematics, will have an effect which is hard to estimate. They favour of course those who have done a good deal of Advanced Mathematics. Such candidates are now pretty safe, while the straightforward easy paper might have ploughed them for inaccuracy. Or again they help an opposite type: the boy who knows that he can do so-and-so and cannot do such-and-such, and who does the so-and-so well, gets 35% on the actual paper as marked and has this raised because of the low totals.

The too hard paper is discouraging for next year's candidates. It also makes the pass mark rather more fluky, because if the marks have to be raised, a few marks fluked make more difference. The "trying for credit" people are less affected. Too easy a paper makes the "very good" class fluky. In fact, of course, the perfect paper should not be too tricky or too hard or worded in a manner too little familiar and should not go beyond the syllabus, but please, teachers, remember that even if a paper has got all these faults, it does not mean that the lists of those who fail to get credit or who are ploughed are any longer. To remember this will save you much unnecessary unhappiness.

What does the examiner ask of the teacher? Primarily, I think, to be regarded as a colleague rather than as an enemy.

A teacher who wishes to criticise an examination paper or a question in it has at least three perfectly satisfactory ways of doing so. He can write direct to one of the examiners, he can write to the secretary of the examination, or he can write to the committee of the Secondary Schools Association which regularly criticises the examination. But if he stands up before a meeting of teachers such as this, let him try to treat the examiner as an "erring colleague" rather than as an "inevitable foe". Alas, teachers try to train their pupils to "defeat the examiner"; and if you try to defeat someone he must be an enemy. So, led away by their own metaphors, it is as an enemy, I fear, that some teachers regard him.

An example, admirable for my purpose, was provided in the debate on a Higher School Certificate syllabus last year by a speaker who severely castigated the board of examiners for London Higher Certificate. Two of the four concerned, I may say, were present and listened in silence, so perhaps if last year's critic is present he will return the compliment. He was speaking of a pair of questions in which forces were represented by lines and the moments of those forces by areas, and, perhaps quite properly, described them as unduly tricky for the papers in which they were included. As far as the substance of his criticism went, he may—possibly—have been justified and I hope that what he said did the examiners good. But I wish to call your attention to two points in the *manner* in which he made his criticism. He was speaking, or was supposed to be speaking about the newly suggested syllabus for Higher Certificate and (I quote from the *Gazette*) used this phrase: "the difficulties that may arise, even with a progressive syllabus such as this, when it is up against

a really determined examiner." While sympathising with the speaker's desire to find at all costs a trenchant phrase, I must ask you to consider the words " a really determined examiner ".

" Determined " to do what?

" To make the examination harder than it should be? " If he does, he knows he is in for trouble in two ways. Not only will the teachers' association reprove him, but a paper that is too hard is far more trouble to mark than a paper that is too easy.

" To ignore the latest changes in the syllabus? " You don't know how sick he is of trying to keep to the old syllabus. If there have been recent changes, the examiner very likely suggested them.

" To show off his own cleverness at the expense of the candidates? " He knows only too well that cleverness as an examiner is totally different from cleverness as a mathematician.

I confess I don't know what the examiner is " determined " about, except perhaps to do his job reasonably well with the least friction and the least trouble to himself. But you see the critic did not speak of " careless " examiners or of " lazy " examiners but of " determined " examiners. That is, he was regarding them as natural enemies. Well, examiners do not like that attitude.

Then there is a smaller point. The same speaker, going out of his way to provide almost the perfect example of how not to criticise examiners, gave another instance of what to avoid. I quote again : " Of the many candidates I presented only those secured the solution who knew that the three diagonals (of a quadrilateral) had collinear midpoints. This solution takes about three lines, and was not what the examiner required." How did he know? It looks like omniscience. Let me try a little omniscience also and assure my audience that this so-called unwanted solution was one of three equally good, equally marked and nearly equally easy solutions which had gone the round of the four examiners concerned long before the paper was printed.

Please, teachers, do not tell your pupils " What I have shown you is the best way of getting the result, but it won't do in an examination." It will. If it is *very* short because it assumes a lot that perhaps should be proved, it may not get quite full marks, but it will get nearly full marks and the time saved will more than make up for the marks lost. So why worry?

One last word about the same speaker, whose words I have been quoting. His arrows, whether blunt or sharp, were directed to the wrong mark. He forgot the moderator. It is the business of the moderator to see that examiners (whether careless or determined) set nothing that is outside the syllabus or otherwise unsuitable for the candidates concerned. So the moderator not only shields the candidates from the careless examiner but also shields the examiner from his critics. It is perhaps an odd thing that the criticisms of that April afternoon should have been unintentionally directed towards the same person whose work in preparing the most up-to-date syllabus for School Certificate has been praised with most cordial unanimity. The moderator's influence in keeping the examiners within the syllabus and in preventing the setting of unsuitable questions should not be minimised, and London examiners have been singularly fortunate when such men as the late Principal Hatton and the very much alive Professor Jeffery undertake the tiresome task of moderating their papers.

This business of keeping to a syllabus is an extremely difficult one. It is as a rule impossible to know what the syllabus for elementary mathematics excludes until one has studied the syllabus for additional mathematics. It may be assumed that what is mentioned as additional mathematics is not elementary ; but there are plenty of things which neither syllabus mentions

but which are ordinarily taught and will naturally be examined somewhere. The question is where? Here is an instance. In the Elementary Mathematics syllabus of the General School Examination (London) appear the words "Arithmetic and Geometric Series". ("Series" here means "progression".) No further allusion to these is included in Mathematics More Advanced. Now where does harmonic progression come in? Anyone who thinks that harmonic progression can be left out of mathematics is deluding himself. Every time he draws a complete quadrilateral and its diagonals, almost innumerable harmonic progressions make a magical appearance. Well! Is an harmonic progression elementary mathematics or mathematics more advanced? This is one of the matters settled by the tradition of the examination. Take another example. The syllabus for Higher Certificate Group III contains the phrase "Elementary three-dimensional geometry, including that of the sphere". Now I personally consider that congruent triangles are one of the most elementary items in geometry and that the phrase quoted must naturally include "congruent triangles on the sphere". So I wanted to set these Group III mathematical specialists the question "Explain why there is a three-angles case of congruence for spherical triangles". One of my colleagues said this was "spherical trigonometry". I said it was not. Where are the sines and cosines? So we agreed that I should put it to the moderator. Well, he turned my question down, his main reason being that the words in the syllabus were "geometry *of* the sphere" and not "geometry *on* the sphere". On such questions of a single letter turn the interpretations of syllabus as in the old days did the interpretations of creeds.

Another difficulty which examiners face is how to get variety of questions on a given syllabus without gradually increasing their difficulty. If the syllabus is a new one it is comparatively easy to prepare an attractive set of papers on it. It is not nearly so easy when the previous three or four years have used up a great many of the most obviously suitable questions. No doubt many of them can be repeated, but questions repeated, having been done in practice papers by most candidates, are never quite the same as they were. Small variations and additions must be devised, and very possibly these additions make the papers harder. It is easy to run them beyond the syllabus accidentally and here is one place where the moderator comes in.

Finally, what is needed to improve the relations between teachers and examiners? Surely it is that they should get together more often. It will not be possible for all teachers, even if they have marked papers at large examinations, to become also setters of papers. There are not enough of these jobs to go round. But machinery should be devised by which teachers can tell examiners *beforehand* the types of question of which they approve and the types of which they disapprove and can settle with the examiners which minor unmentioned items are included among the major items of the syllabus. At present the time-lag is very considerable, and it is even worse as to changes of syllabus since teachers require such long notice of any changes. In Mr. Brereton's book, *The Case for Examinations*, he discusses this matter at greater length than I can do now. One of his conclusions, with which I heartily agree, mentions as a step immediately possible :

"The examining bodies could arrange regular meetings of the teachers in each subject in the schools taking their respective examinations. The examiners would be present at such meetings and discussions would take place on past and future examinations."

One last word. If the General School Examination ever becomes an internal one, as the Norwood Report would have it, many more teachers will have brought home to them the difficulties of examiners when the results of the

examination are important outside the school. If they have been scornful about examiners, perhaps they will then see reasons to change their attitude.

C. O. T.

## NEW SOUTH WALES BRANCH.

### Report for 1944.

The membership of this Branch is now 19, with 124 associates. During the year, the following addresses were given.

(i) Mr. R. J. Gillings gave an address entitled " $2^{64}$ ". Some interesting side-lights on the history of mathematics were given. It was suggested by the speaker and others during the succeeding discussion that class-room work could benefit from the introduction of historical references in many of the topics dealt with.

(ii) Mr. H. J. Meldrum put forward some suggestions about the introduction of new topics into the general course of the secondary school. It was suggested that the basis for any school course was well expressed in the Hadow Report. Possible new topics might be some of the elements of statistics, and enough spherical geometry and trigonometry to deal with simple problems of the measurement of time, of navigation and elementary astronomy. Mr. Meldrum made a number of suggestions about the topics at present in the syllabus that might be omitted; the suggestions were mainly concerned with the pruning of mechanical work from the present syllabus.

(iii) Mr. Hone of Cranbrook School spoke of his experiences as a master, who took some part in the mathematical work at Marlborough at one stage of his career. This proved one of the most interesting talks of the year.

(iv) Professor E. M. Wellish gave an address entitled " Mathematical topics appropriate to School and University courses ". Professor Wellish outlined the treatment of several topics showing how school work could be expanded into fuller treatment of the same topic at the University.

The most important decision of the Branch made in 1944 was one dealing with the desirability of the publication of a journal to be distributed throughout Australia. The first number will be published about April, 1945. Among other purposes that the journal will serve will be the publication of addresses for the benefit of those unable to attend the meetings of the various Branches.

The officers for 1945 were elected as follows: *President*: Mr. P. G. Price; *Joint Hon. Secretaries*: Miss Ida Barnes, Mr. H. J. Meldrum; *Hon. Treasurer and Director of the Problem Bureau*: Mr. R. J. Gillings.

Ida Barnes, H. J. Meldrum, *Joint Hon. Secretaries.*

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**1453.** The peak of Utopia is steep; the serpentine-road which leads up to it has many tortuous curves. While you are moving up the road you never face the peak, your direction is the tangent, leading nowhere. If a great mass of people is pushing forward along the serpentine they will, according to the fatal laws of inertia, push off their leader from the road and then follow him, the whole movement flying off at the tangent into the nowhere. That is what happened to most revolutionary movements, where the mass-impulse is strong and the inertia of the mass is converted into a violent centrifugal force. In the more cautious reformist movements, on the other hand, the momentum soon fades out and the ascending spiral first becomes a wiry circling round and round the peak without gaining in height until it finally degenerates into a descending spiral; *e.g.* the Trade Unionist movement.—Arthur Koestler, " The Yogi and the Commissar ", *Horizon*, June 1942. [Per Mr. D. J. Price.]

# THE MATHEMATICAL THEORY OF THE INFLUENCE OF THIN FILMS ON THE REFLECTION AND TRANSMISSION OF LIGHT.

By J. Deans.

If white light falls on a soap bubble or on a film of oil on a pool of water, the reflected light is very often coloured, due to interference at the surface of the film. Within the last few years fresh interest has been added to the phenomenon by the work of John Strong, Katherine B. Blodgett and others, who have shown that interference films can be deposited on glass and that these films reduce the reflection from the surface and increase the transmission through it, a matter of great importance in optical instruments. In many of the writings on the subject, the intensity of the reflected beam only has been considered, the transmitted beam being dismissed with the statement that, by the Conservation of Energy, light which is not reflected is transmitted. In some of the writings, only perpendicular incidence has been considered. In the discussion below expressions are obtained for the reflected and transmitted beams without the use of the Conservation of Energy for all angles of incidence.

*The Statement of the Problem.*

Let two transparent media of refractive indices $n_0$ and $n_1$ be separated by a thin film of refractive index $\mu$, the surfaces of the film being plane and parallel. It is assumed that no absorption takes place in the film. Let a plane polarised plane wave of monochromatic light be incident on the first surface of the film—the $n_0\mu$ surface—in the direction $AB$ in the medium $n_0$ (Fig. 1). At $B$ on the $n_0\mu$ surface the wave is partly reflected in the direction $BE$ and partly transmitted in the direction $BC$. At $C$ on the $\mu n_1$ surface it

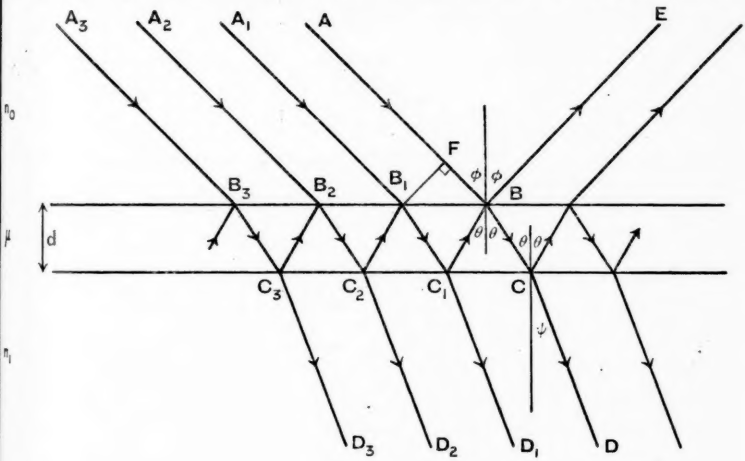


Fig. 1.

is again divided into another pair of reflected and transmitted waves. Whenever a wave meets the $n_0\mu$ or the $\mu n_1$ surface, it is partly reflected and partly

transmitted. The resultant reflected and transmitted waves are therefore built from waves which have travelled paths of different lengths in the film and which are of different intensities according to the number of reflections they have undergone.

Let the components of the electric intensities of the incident, resultant reflected and transmitted waves in the plane of incidence be represented by

$$I \sin \frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right), \quad R \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right) - \Delta_R\right]$$

and

$$A \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_1 x}{c}\right) - \Delta_A\right],$$

and perpendicular to the plane of incidence by

$$i \sin \frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right), \quad r \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right) - \delta_r\right]$$

and

$$a \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_1 x}{c}\right) - \delta_a\right]$$

respectively, where $\tau$ and $c$ represent the period and the velocity of the wave *in vacuo*, and $t$ and $x$ are measures of time and distance along the path of the wave. By Maxwell's Equations, the components of the magnetic intensities of these waves in the plane of incidence are represented by

$$-n_0 i \sin \frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right), \quad -n_0 r \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right) - \delta_r\right].$$

and

$$-n_1 a \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_1 x}{c}\right) - \delta_a\right],$$

and perpendicular to the plane of incidence by

$$n_0 I \sin \frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right), \quad n_0 R \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_0 x}{c}\right) - \Delta_R\right]$$

and

$$n_1 A \sin\left[\frac{2\pi}{\tau}\left(t - \frac{n_1 x}{c}\right) - \Delta_A\right].$$

It is required to find mathematical expressions for the amplitudes $R$, $A$, $r$ and $a$ and the phase differences $\Delta_R$, $\Delta_A$, $\delta_r$ and $\delta_a$ in terms of the amplitudes $I$ and $i$ of the incident wave and of the other constants of the media and the wave.

*Notation.*

Convenient constants are $d$, the thickness of the film; $\phi$, the angle of incidence of the wave; $\theta$, the angle of refraction in the film; $\psi$, the angle of refraction in the medium $n_1$; and quantities $R_0$, $R_0'$, $R_1$, $A_0$, $A_0'$, $A_1$, $r_0$, $r_0'$, $r_1$, $a_0$, $a_0'$ and $a_1$, which are defined thus: Let a plane wave of amplitude $p$ be incident at the angle $\phi$ on the $n_0\mu$ surface from the medium $n_0$; then if the wave is polarised perpendicular to the plane of incidence (*i.e.* if the electric intensity is in the plane of incidence), the amplitudes of the reflected and refracted waves are $R_0 p$ and $A_0 p$ respectively, and if the wave is polarised in the plane of incidence (*i.e.* if the electric intensity is perpendicular to the plane of incidence), the amplitudes of the reflected and refracted waves are $r_0 p$ and $a_0 p$. If the wave is incident from the film at the angle $\theta$ on the $n_0\mu$ surface, the corresponding quantities are $R_0'p$, $A_0'p$, $r_0'p$ and $a_0'p$, and if the wave is incident from the film at the angle $\theta$ on the $\mu n_1$ surface, they are

$R_1p$, $A_1p$, $r_1p$ and $a_1p$. By Fresnel's Formulae, as derived from the Electromagnetic Theory of Light,

$$R_0 = \frac{\tan(\phi-\theta)}{\tan(\phi+\theta)}, \qquad A_0 = \frac{2\sin\theta\cos\phi}{\sin(\phi+\theta)\cos(\phi-\theta)},$$

$$R_0' = \frac{\tan(\theta-\phi)}{\tan(\theta+\phi)}, \qquad A_0' = \frac{2\sin\phi\cos\theta}{\sin(\theta+\phi)\cos(\theta-\phi)},$$

$$R_1 = \frac{\tan(\theta-\psi)}{\tan(\theta+\psi)}, \qquad A_1 = \frac{2\sin\psi\cos\theta}{\sin(\theta+\psi)\cos(\theta-\psi)}, \quad \text{............(1)}$$

$$r_0 = -\frac{\sin(\phi-\theta)}{\sin(\phi+\theta)}, \qquad a_0 = \frac{2\sin\theta\cos\phi}{\sin(\phi+\theta)},$$

$$r_0' = -\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)}, \qquad a_0' = \frac{2\sin\phi\cos\theta}{\sin(\theta+\phi)},$$

$$r_1 = -\frac{\sin(\theta-\psi)}{\sin(\theta+\psi)}, \qquad a_1 = \frac{2\sin\psi\cos\theta}{\sin(\theta+\psi)}.$$

These give

$$R_0' = -R_0, \quad A_0A_0' + R_0^2 = 1, \quad r_0' = -r_0 \quad \text{and} \quad a_0a_0' + r_0^2 = 1. \text{ ......(2)}$$

Let $\lambda$ be the wavelength of the light *in vacuo*; then

$$\lambda = c\tau.$$

*The Determination of $R$, $A$, $r$, $a$, $\Delta_R$, $\Delta_A$, $\delta_r$ and $\delta_a$.*

Consider now the resultant reflected wave along $BE$ (Fig. 1). Let us take first the component of the electric intensity in the plane of incidence. The successive waves which form the resultant reflected wave at $B$ have travelled along

$$AB, \quad A_1B_1C_1B, \quad A_2B_2C_2B_1C_1B, \quad A_3B_3C_3B_2C_2B_1C_1B, \ldots\ldots\ldots$$

and therefore have amplitudes

$$R_0I, \quad A_0A_0'R_1I, \quad A_0A_0'R_0'R_1^2I, \quad A_0A_0'R_0'^2R_1^3I, \ldots\ldots\ldots$$

There is also a constant phase difference between successive waves of the series due to the constant path difference between them. Let $B_1F$ be perpendicular to $AB$, meeting $AB$ at $F$. Then the constant optical path difference is equal to

$$\begin{aligned} &\mu B_1C_1 + \mu C_1B - n_0FB \\ &= 2\mu d\sec\theta - 2n_0d\sin\phi\tan\theta \\ &= 2\mu d\cos\theta \quad \text{since} \quad n_0\sin\phi = \mu\sin\theta. \end{aligned}$$

Since the waves along $A_1B_1$ and $AB$ are in phase along $B_1F$, the constant phase difference is therefore $(4\pi\mu d\cos\theta)/\tau c$, which is equal to $(4\pi\mu d\cos\theta)/\lambda$. Let this phase difference be denoted by $\delta$.

The electric intensity of the resultant reflected wave in the plane of incidence is therefore represented by

$$R\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)-\Delta_R\right] = R_0I\sin\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)$$
$$+ A_0A_0'R_1I\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)-\delta\right]$$

$$+A_0A_0'R_0'R_1^2I\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)-2\delta\right]$$

$$+A_0A_0'R_0'^2R_1^3I\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)-3\delta\right]$$

$$+\dots\dots\dots .$$

Since this equation is true for all values of $\left(t-\frac{n_0x}{c}\right)$, the trigonometrical functions may be expanded by the addition formula and the coefficients of $\sin\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)$ and $\cos\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)$ equated.

Thus $\quad R\cos\Delta_R=R_0I+A_0A_0'R_1I\cos\delta+A_0A_0'R_0'R_1^2I\cos 2\delta$

$$+A_0A_0'R_0'^2R_1^3I\cos 3\delta+\dots,$$

and $\quad R\sin\Delta_R=\quad A_0A_0'R_1I\sin\delta+A_0A_0'R_0'R_1^2I\sin 2\delta$

$$+A_0A_0'R_0'^2R_1^3I\sin 3\delta+\dots .$$

On multiplying the second equation by $\sqrt{(-1)}$ and adding to the first, there is obtained on the right-hand side after the term $R_0I$ a geometric progression which can be summed. When $R_0'$ and $A_0'$ are removed by (2) and the real and imaginary parts are separated, it is found that

$$R\cos\Delta_R=I\cdot\frac{R_0(1+R_1^2)+R_1(1+R_0^2)\cos\delta}{1+2R_0R_1\cos\delta+R_0^2R_1^2}$$

and

$$R\sin\Delta_R=I\cdot\frac{R_1(1-R_0^2)\sin\delta}{1+2R_0R_1\cos\delta+R_0^2R_1^2}. \quad\dots\dots\dots\dots(3)$$

Thus

$$R^2=I^2\cdot\frac{R_0^2+2R_0R_1\cos\delta+R_1^2}{1+2R_0R_1\cos\delta+R_0^2R_1^2}$$

and

$$\tan\Delta_R=\frac{R_1(1-R_0^2)\sin\delta}{R_0(1+R_1^2)+R_1(1+R_0^2)\cos\delta}.$$

The use of the component of the magnetic intensity perpendicular to the plane of incidence clearly leads to the same expressions for $R^2$ and $\Delta_R$.

In the same way it may be shown that for the component polarised in the plane of incidence,

$$r\cos\delta_r=i\cdot\frac{r_0(1+r_1^2)+r_1(1+r_0^2)\cos\delta}{1+2r_0r_1\cos\delta+r_0^2r_1^2},$$

$$r\sin\delta_r=i\cdot\frac{r_1(1-r_0^2)\sin\delta}{1+2r_0r_1\cos\delta+r_0^2r_1^2}, \quad\dots\dots\dots\dots(3)$$

$$r^2=i^2\cdot\frac{r_0^2+2r_0r_1\cos\delta+r_1^2}{1+2r_0r_1\cos\delta+r_0^2r_1^2},$$

and

$$\tan\delta_r=\frac{r_1(1-r_0^2)\sin\delta}{r_0(1+r_1^2)+r_1(1+r_0^2)\cos\delta}.$$

Similarly, to find expressions for $A^2$ and $\Delta_A$, it is necessary to consider only the component of the electric intensity of the transmitted wave along $CD$ in the plane of incidence. Therefore,

$$A\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_1x}{c}\right)-\Delta_A\right]=A_0A_1I\sin\frac{2\pi}{\tau}\left(t-\frac{n_1x}{c}\right)$$

$$+A_0A_1R_0'R_1I\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_1x}{c}\right)-\delta\right]$$

$$+A_0A_1R_0'^2R_1^2I\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_1x}{c}\right)-2\delta\right]$$

$$+A_0A_1R_0'^3R_1^3I\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_1x}{c}\right)-3\delta\right]$$

$$+\ldots\ldots\ldots,$$

which gives
$$A\cos\Delta_A=\ .\ I\cdot\frac{A_0A_1(1+R_0R_1\cos\delta)}{1+2R_0R_1\cos\delta+R_0^2R_1^2},$$

$$A\sin\Delta_A=-I\cdot\frac{A_0A_1R_0R_1\sin\delta}{1+2R_0R_1\cos\delta+R_0^2R_1^2},$$

$$A^2=I^2\ \cdot\frac{A_0^2A_1^2}{1+2R_0R_1\cos\delta+R_0^2R_1^2},$$

and
$$\tan\Delta_A=\ -\frac{R_0R_1\sin\delta}{1+R_0R_1\cos\delta}.$$

Similarly, for the component polarised in the plane of incidence, it may be shown that

$$a\cos\delta_a=\ i\cdot\frac{a_0a_1(1+r_0r_1\cos\delta)}{1+2r_0r_1\cos\delta+r_0^2r_1^2},$$

$$a\sin\delta_a=-i\cdot\frac{a_0a_1r_0r_1\sin\delta}{1+2r_0r_1\cos\delta+r_0^2r_1^2},$$

$$a^2=\ i^2\cdot\frac{a_0^2a_1^2}{1+2r_0r_1\cos\delta+r_0^2r_1^2},$$

and
$$\tan\delta_a=\ -\frac{r_0r_1\sin\delta}{1+r_0r_1\cos\delta}.$$

*The Polarisation of the Reflected and Transmitted Waves.*

Let rectangular axes $OX$, $OY$ and $OZ$ be taken such that $OX$ is the path of the wave under consideration, $OY$ lies in the plane of incidence and $OZ$ is perpendicular to the plane of incidence.

Let $y=R\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)-\Delta_R\right]$ and $z=r\sin\left[\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)-\delta_r\right]$.

In general $R\neq r$ and $\Delta_R\neq\delta_r$. The elimination of $\frac{2\pi}{\tau}\left(t-\frac{n_0x}{c}\right)$ between these equations results in

$$r^2y^2+R^2z^2-2Rryz\cos(\Delta_R-\delta_r)=R^2r^2\sin^2(\Delta_R-\delta_r),\ \ldots\ldots\ldots\ldots(4)$$

an ellipse, which is swept out by the vector representing the resultant electric intensity of the reflected wave, which is therefore constant neither in magnitude nor in direction. The reflected wave is elliptically polarised. Similarly, the transmitted wave also is elliptically polarised.

Two cases are of special interest, that in which $\sin\delta=0$ and that in which the light is incident perpendicularly on the film.

*Case I.* When $\sin \delta = 0$, $\cos \delta = \pm 1$, $\sin (\Delta_R - \delta_r) = 0$ and $\cos (\Delta_R - \delta_r) = \pm 1$, the signs depending on the values of the constants of the wave and of the media. In numerical work these signs are easily found from the formulae (3) for $R \cos \Delta_R$ and $r \cos \delta_r$. The ellipse degenerates to part of one or other of the straight lines given by $ry \mp Rz = 0$. The reflected wave is then plane polarised. Similarly, when $\sin \delta = 0$, the transmitted wave is plane polarised.

The condition $\sin \delta = 0$ can be written as $(4\pi\mu d \cos \theta)/\lambda = k\pi$, *i.e.* $\mu d \cos \theta = \frac{1}{4}k\lambda$, where $k$ is any positive integer.

*Case II.* With perpendicular incidence any plane through the path of the wave can be regarded as the plane of incidence. If this is taken as the plane of the electric intensity, then it is obvious that the reflected and transmitted waves are plane polarised in the same plane as the incident wave. This fact can, however, be deduced from the general analysis thus :

Let $\phi \to 0$ in (1). Then

$$R_0 \to \frac{\mu - n_0}{\mu + n_0}, \qquad r_0 \to -\frac{\mu - n_0}{\mu + n_0},$$

$$R_1 \to \frac{n_1 - \mu}{n_1 + \mu}, \qquad r_1 \to -\frac{n_1 - \mu}{n_1 + \mu},$$

$$A_0 \to \frac{2n_0}{\mu + n_0}, \qquad a_0 \to \frac{2n_0}{\mu + n_0},$$

$$A_1 \to \frac{2\mu}{n_1 + \mu}, \qquad a_1 \to \frac{2\mu}{n_1 + \mu}.$$

Hence
$$(R/I) \cos \Delta_R = -(r/i) \cos \delta_r,$$
and
$$(R/I) \sin \Delta_R = -(r/i) \sin \delta_r.$$

The ellipse (4) then reduces to part of the straight line given by $y/z = -I/i$, that is, the resultant reflected wave is plane polarised in the same plane as the incident wave. The negative sign is explained by the turning of the axes with the wave on reflection. Similarly, the transmitted wave is plane polarised in the same plane.

*The Maximum and Minimum Values of $R^2$, $r^2$, $A^2$ and $a^2$.*

Since the intensities of the reflected and transmitted waves are proportional to $R^2$, $r^2$, $A^2$ and $a^2$, according as the components are polarised perpendicular to or in the plane of incidence, it is of some interest to consider the variation of these functions with $d$, the thickness of the film. The angle of incidence and the refractive indices of the film and the media are constant throughout this discussion.

$A^2$ and $a^2$ have turning values when $\cos \delta = \pm 1$. Since

$$\frac{d(R^2)}{d\delta} = -I^2 \frac{2R_0R_1(1 - R_0^2)(1 - R_1^2) \sin \delta}{(1 + 2R_0R_1 \cos \delta + R_0^2R_1^2)^2}$$

and
$$\frac{d(r^2)}{d\delta} = -i^2 \frac{2r_0r_1(1 - r_0^2)(1 - r_1^2) \sin \delta}{(1 + 2r_0r_1 \cos \delta + r_0^2r_1^2)^2},$$

and since $d(R^2)/d\delta$ and $d(r^2)/d\delta$ change sign as $\sin \delta$ passes through zero, $R^2$ and $r^2$ also have turning values when $\sin \delta = 0$, *i.e.* when $\cos \delta = \pm 1$.

When $\mu$ lies between $n_0$ and $n_1$, $\frac{1}{2}\pi > \phi > \theta > \psi$ or $\phi < \theta < \psi < \frac{1}{2}\pi$, and hence $R_0R_1$ and $r_0r_1$ are positive by (1). In this case $\cos \delta = -1$ leads to a minimum value of $R^2$ and a maximum value of $A^2$, and $\cos \delta = +1$ to a maximum value

of $R^2$ and a minimum value of $A^2$. If $\cos\delta = -1$, $(4\pi\mu d\cos\theta)/\lambda = (2k-1)\pi$, and therefore $\mu d\cos\theta = (2k-1)\lambda/4$, where $k$ is any positive integer, and if $\cos\delta = +1$, $\mu d\cos\theta = k\lambda/2$. If the light is incident perpendicularly, $\cos\theta = 1$, and the minimum value of $R^2$ and the maximum value of $A^2$ occur when $\mu d$, the optical thickness of the film, is an odd number of quarter-wavelengths of the light, and the maximum value of $R^2$ and the minimum value of $A^2$ when the optical thickness of the film is an even number of quarter-wavelengths. The same result is true for $r^2$ and $a^2$.

When $\mu$ does not lie between $n_0$ and $n_1$, $R_0R_1$ and $r_0r_1$ are negative and therefore $\cos\delta = -1$ leads to maximum values of $R^2$ and $r^2$ and minimum values of $A^2$ and $a^2$, and $\cos\delta = +1$ to minimum values of $R^2$ and $r^2$ and maximum values of $A^2$ and $a^2$.

The following table gives expressions for the maxima and minima in the various cases :

| Function | Nature of turning value | Turning value | Condition for turning value | Turning value at perpendicular incidence |
|---|---|---|---|---|
| *$\mu$ between $n_0$ and $n_1$.* | | | | |
| $R^2$ | Maximum | $I^2\left(\dfrac{R_0+R_1}{1+R_0R_1}\right)^2$ | $\mu d\cos\theta = k\lambda/2$ | $I^2\left(\dfrac{n_1-n_0}{n_1+n_0}\right)^2$ |
| | Minimum | $I^2\left(\dfrac{R_0-R_1}{1-R_0R_1}\right)^2$ | $\mu d\cos\theta = (2k-1)\lambda/4$ | $I^2\left(\dfrac{\mu^2-n_0n_1}{\mu^2+n_0n_1}\right)^2$ |
| $A^2$ | Maximum | $I^2\left(\dfrac{A_0A_1}{1-R_0R_1}\right)^2$ | $\mu d\cos\theta = (2k-1)\lambda/4$ | $I^2\left(\dfrac{2n_0\mu}{\mu^2+n_0n_1}\right)^2$ |
| | Minimum | $I^2\left(\dfrac{A_0A_1}{1+R_0R_1}\right)^2$ | $\mu d\cos\theta = k\lambda/2$ | $I^2\left(\dfrac{2n_0}{n_1+n_0}\right)^2$ |
| *$\mu$ not between $n_0$ and $n_1$.* | | | | |
| $R^2$ | Maximum | $I^2\left(\dfrac{R_0-R_1}{1-R_0R_1}\right)^2$ | $\mu d\cos\theta = (2k-1)\lambda/4$ | $I^2\left(\dfrac{\mu^2-n_0n_1}{\mu^2+n_0n_1}\right)^2$ |
| | Minimum | $I^2\left(\dfrac{R_0+R_1}{1+R_0R_1}\right)^2$ | $\mu d\cos\theta = k\lambda/2$ | $I^2\left(\dfrac{n_1-n_0}{n_1+n_0}\right)^2$ |
| $A^2$ | Maximum | $I^2\left(\dfrac{A_0A_1}{1+R_0R_1}\right)^2$ | $\mu d\cos\theta = k\lambda/2$ | $I^2\left(\dfrac{2n_0}{n_1+n_0}\right)^2$ |
| | Minimum | $I^2\left(\dfrac{A_0A_1}{1-R_0R_1}\right)^2$ | $\mu d\cos\theta = (2k-1)\lambda/4$ | $I^2\left(\dfrac{2n_0\mu}{\mu^2+n_0n_1}\right)^2$ |

If the capital letters are replaced by small letters in the above table, the expressions for the turning values of $r^2$ and $a^2$ are obtained.

It may be shown by substitution from (1) that

$$I^2\left(\frac{R_0+R_1}{1+R_0R_1}\right)^2 = I^2\frac{\tan^2(\phi-\psi)}{\tan^2(\phi+\psi)},$$

$$i^2\left(\frac{r_0+r_1}{1+r_0r_1}\right)^2 = i^2\frac{\sin^2(\phi-\psi)}{\sin^2(\phi+\psi)},$$

$$I^2\left(\frac{A_0A_1}{1+R_0R_1}\right)^2=I^2\,\frac{4\sin^2\psi\cos^2\phi}{\sin^2(\phi+\psi)\cos^2(\phi-\psi)},$$

and
$$i^2\left(\frac{a_0a_1}{1+r_0r_1}\right)^2=i^2\,\frac{4\sin^2\psi\cos^2\phi}{\sin^2(\phi+\psi)}.$$

The expressions on the right-hand side are Fresnel's Formulae, (1), for the squares of the amplitudes of the reflected and transmitted waves when the media $n_0$ and $n_1$ are not separated by a film. It follows then, that the film weakens the reflection and strengthens the transmission when its refractive index is between $n_0$ and $n_1$, and that it strengthens the reflection and weakens the transmission when its refractive index does not lie between $n_0$ and $n_1$.

When $\mu$ lies between $n_0$ and $n_1$, the coefficient $\left(\dfrac{\mu^2-n_0n_1}{\mu^2+n_0n_1}\right)^2$ in the expression for the minimum value of $R^2$, for perpendicular incidence, is a function of $\mu$, and the minimum intensity of the reflected wave has therefore itself a minimum value, which is zero, when $\mu^2=n_0n_1$. The maximum intensity of the transmitted wave has a corresponding maximum.

*The Conservation of Energy.*

According to the Electromagnetic Theory, the rate of flow of energy across unit area perpendicular to a wavefront is $(nc/8\pi)X^2$ where $n$ is the refractive index of the medium and $X$ the amplitude of the wave.

Select unit area of the $n_0\mu$ surface. Then the cross-sectional areas of those parts of the wave which are incident and reflected there, are both $\cos\phi$. The cross-sectional area of the corresponding part of the transmitted wave in the medium $n_1$ is $\cos\psi$. If the Conservation of Energy holds, it will be expressed by

$$n_0I^2\cos\phi=n_0R^2\cos\phi+n_1A^2\cos\psi$$

and
$$n_0i^2\cos\phi=n_0r^2\cos\phi+n_1a^2\cos\psi, \qquad\qquad (5)$$

when the common factor $c/8\pi$ is removed.

The simplified equations for ordinary reflection and refraction at the $n_0\mu$ and $\mu n_1$ surfaces,

$$\left.\begin{aligned}n_0\cos\phi&=n_0R_0{}^2\cos\phi+\mu A_0{}^2\cos\theta\\ \mu\cos\theta&=\mu R_1{}^2\cos\theta+n_1A_1{}^2\cos\psi.\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned}n_0\cos\phi&=n_0r_0{}^2\cos\phi+\mu a_0{}^2\cos\theta\\ \mu\cos\theta&=\mu r_1{}^2\cos\theta+n_1a_1{}^2\cos\psi\end{aligned}\right\},$$

are easily verified by (1). If $\theta$ is eliminated between each pair of these equations, the relations (5) are obtained. The Conservation of Energy therefore is true for these waves.

*Efficiency with White Light.*

Since the thickness of the film for maximum efficiency in the reduction of reflection and increase of transmission depends on the wavelength of the light, no film is equally effective over the range of wavelengths in the visible spectrum. The curves in Fig. 2 show the variation with the thickness of the film of the intensity of the light reflected from a film of refractive index 1·3 on a crown glass of refractive index 1·5 for perpendicular incidence in air. They are drawn for the spectral lines 4047 A.U., 5461 A.U. and 6563 A.U., which are in the violet, green and red parts of the spectrum. The line representing the reflectivity of an unfilmed surface of the same glass is also shown. The effect of the dispersions of the glass and of the film is small and is neglected, since no data of the dispersions of these films are available. It is

clear from the curves that for white light greatest efficiency in the reduction of reflection and increase of transmission is attained when the optical thickness of the film is about 1350 A.U., *i.e.* one-quarter of the wavelength of

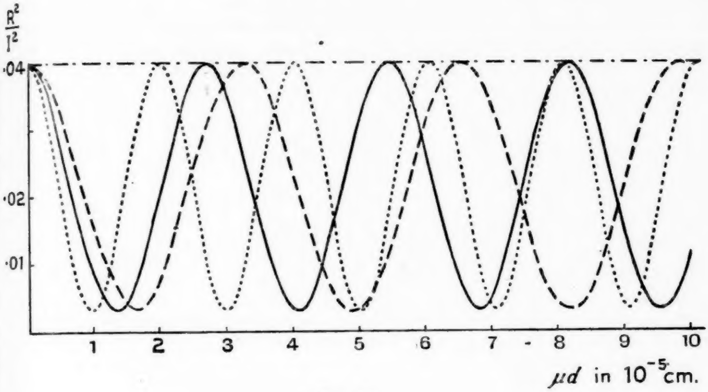


Fig. 2.

$n_0 = 1.$ $\mu = 1{\cdot}3.$ $n_1 = 1{\cdot}5.$

...... Violet 4047 A.U. — — — Red 6563 A.U.

—— Green 5461 A.U. —·—·— Unfilmed surface.

green light. The differences in the reflectivities of the components of white light result in the reflected light being coloured, but the transmitted light is not visibly coloured as the differences in the transmission of the various wavelengths are small compared with their intensities.

In an optical instrument two advantages are generally claimed for filming the glass surfaces. The increase in the transmission gives a brighter field, and the reduction in the internal reflections diminishes the general background flare which destroys contrast. The substances of which the films are made are usually fluorides of the alkali and alkaline earth metals. Particulars of these and of the preparation and other properties of the films are given in the papers listed below. J. D.


References.

*For Miscellaneous Optical Formulae.*

R. A. Houstoun, *Treatise on Light.*

*For the Optical Properties of Thin Films.*

K. B. Blodgett, *Phys. Rev.*, Vol. 55, p. 391 (1939).

K. M. Greenland, *Nature*, Vol. 152, p. 290 (1943).


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**1454.** He was an arithmetician rather than a mathematician. None of the humour, the music or the mysticism of higher mathematics ever entered his head. Men might vary in height or weight or colour, just as 6 is different from 8, but there was little other difference.—John Steinbeck, *The Moon is Down* (1942). [Per Mr. J. W. Ashley Smith.]

# ON THE THREE-CUSPED HYPOCYCLOID.

By J. Hadamard.

I have had a recent opportunity to recall an early article (1884) which I wrote on the three-cusped hypocycloid. My starting point was the property that the asymptotes of any pencil of equilateral hyperbolas envelop such a hypocycloid.* I proved this analytically in the aforesaid article; perhaps there is some interest in finding geometrical reasons for it.

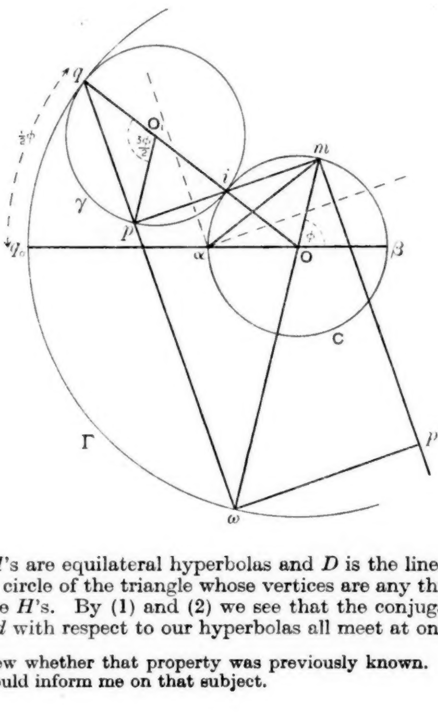
Principles on pencils of conics are well known. According to these principles:

(1) The polars of any point $a$ with respect to the various conics of the pencil are concurrent at one and the same point $\alpha$, which we shall call the corresponding point of $a$.

(2) If $a$ describes a straight line $D$, then $\alpha$ describes a certain conic $C$.

(3) This conic $C$ is also the locus of the poles of $D$ with respect to the conics of the pencil, a consequence being:

(4) If $m$, a point of $C$, is the pole of $D$ with respect to one of the conics $H$ of the pencil and $a$ a point of $D$ with the corresponding point $\alpha$, then the polar line of $a$ with respect to $H$ is $m\alpha$.




When the $H$'s are equilateral hyperbolas and $D$ is the line at infinity, $C$ is the nine-point circle of the triangle whose vertices are any three of the points common to the $H$'s. By (1) and (2) we see that the conjugate diameters of any direction $d$ with respect to our hyperbolas all meet at one point $\alpha$, which

* I never knew whether that property was previously known. I should be glad if any reader could inform me on that subject.

is a point of $C$. As $d$ and its conjugate direction rotate with equal but opposite angular velocities, we can choose $d$ so that the corresponding radius of $C$ goes through the point $\beta$ diametrically opposite to $\alpha$, an unimportant assumption made only in order to simplify notation.

This being said, let us consider any one of our hyperbolas $H$, the centre of which is the point $m$ of $C$, deduced from $\beta$ by a rotation of angle $\phi$ round the centre $O$ of $C$. On account of (4), the conjugate diameter of $d$ with respect to this $H$ will be $\alpha m$. Therefore, *the asymptotic directions for this $H$ will be the lines bisecting the angles between $d$ and $\alpha m$*, one of which will cut $d$ at the angle $\phi/4$, the other at that angle increased * by $\pi/2$. Parallels to these directions through $m$ will give the asymptotes themselves.

Let us consider the motion of the plane diagram invariably connected with the right angle between these asymptotes. If $m$ describes $C$, say in the direct sense and with unit angular velocity, its linear velocity will be $l$, the radius of $C$; on the other hand, the angular velocity of the asymptotes will be $\frac{1}{4}$, in the same sense. Therefore the instantaneous centre of rotation must lie on the normal to $C$, that is, on $mO$, prolonged three times beyond $O$. This centre $\omega$ being at a distance $3l$ from $O$ and $4l$ from $m$, we get a kinematic generation of the hypocycloid, not as a locus of points but as an envelope of tangents, viz.:

*When a circle of radius $R$ rolls round a circle of radius $\frac{3}{4}R$ lying inside the rolling circle, a radius vector of the rolling circle envelops a three-cusped hypocycloid.* This, however, is subordinate to our main conclusion, which we now proceed to obtain.

The points of contact of the curve with the two perpendicular tangents from $m$ are the projections $p$, $p'$ of $\omega$ on these lines. Now let $q$ be the second intersection of $\omega p$ with the circle $\Gamma$ of centre $O$ and radius $O\omega = 3l$; if the straight line $Oq$ meets $C$ at $i$, the circle $\gamma$ on $iq$ as diameter will pass through $p$ and touch $\Gamma$, its radius being $l$ (and $i$ being the midpoint of $mp$). The arc $pq$ of $\gamma$ will correspond to a central angle:

$$\angle qO_1p = 2\angle miO = 2\angle Omi = 3\phi/2,$$

while the radius $Oq$ of $\Gamma$ will cut $d$ at an angle

$$\angle qO\omega - \angle \alpha O\omega = 3\phi/2 - \phi = \phi/2.$$

Therefore, if $q_0$ is the point on $\Gamma$ where it is met by the radius $O\alpha$, the two arcs $qp$ and $qq_0$ have the same length, so that the motion of $p$ is generated by the rolling of $\gamma$ inside $\Gamma$. Q.E.D.

We can notice, even without our final argument in the last paragraph above, that it was to be seen that all curves of the kind considered must be similar to each other. Our whole diagram is defined once $C$ and the point $\alpha$ are given, without any further knowledge of the pencil. Especially, it appears that we should not have diminished generality by assuming from the beginning that the four basic points of the pencil were the vertices of an equilateral triangle and its centre, whence the ternary symmetry of the curve ensues.

J. H.

* In what follows, all angular equalities are to be understood as congruences (mod $\pi/2$).

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**1455**. He had seen the whole thing as a triangle—one leg of the triangle stretched from his ridge to the spot where the enemy tanks were now, and the other two legs bisected at the spot where he intended to meet his enemy and offer him battle.—D. E. Stevenson, *The Two Mrs. Abbots*, p. 107. [Per Mr. C. H. Hardingham.]

# HILL'S DIFFERENTIAL EQUATION.

By N. W. McLachlan.

1. We deal with the equation

$$d^2y/dt^2 + \{a - 2k^2\psi(\omega t)\}y = 0, \quad \text{.........................(1)}$$

where $\psi(\omega t)$ is continuous, periodic in $t$ with period $2\pi/\omega$; $a$, $k^2$ are real and positive, and

$$a \gg \omega^2 \geqslant 1, \quad a \gg |2k^2\psi(\omega t)|_{\max}.$$

A stable (bounded) solution was given by Erdélyi in 1934 employing a complicated transformation and integral equations.*

We shall obtain this solution by an elementary method, thereby reducing the analysis to one-third of its former length.

Write $a = \nu^2$, $\sqrt{\{1 - 2k^2\psi(\omega t)/a\}} = \rho$, and (1) becomes

$$d^2y/dt^2 + \nu^2\rho^2 y = 0. \quad \text{.................................(2)}$$

Assume that $y = \exp(\nu \int^t w\, du)$, then $y \neq 0$, and (2) transforms to the Riccati type of equation:

$$\frac{1}{\nu}\frac{dw}{dt} + w^2 + \rho^2 = 0. \quad \text{...............................(3)}$$

Now assume that

$$w = w_0 + (w_1/\nu) + (w_2/\nu^2) + \dots, \quad \text{...................................(4)}$$

$$w^2 = w_0^2 + (2w_0w_1/\nu) + (w_1^2 + 2w_0w_2)/\nu^2 + \dots, \quad \text{.................(5)}$$

the $w$'s being continuous functions † of $t$. Substituting from (4), (5) into (3) gives

$$w_0^2 + \rho^2 + (\dot{w}_0 + 2w_0w_1)/\nu + (\dot{w}_1 + w_1^2 + 2w_0w_2)/\nu^2 + \dots = 0. \quad \text{.............(6)}$$

Equating the coefficients of $\nu^0$, $\nu^{-1}$, $\nu^{-2}$, ... to zero, yields the following:

$$w_0^2 + \rho^2 = 0, \quad \text{...............................(7a)}$$

$$\dot{w}_0 + 2w_0w_1 = 0, \quad \text{...........................(7b)}$$

$$\dot{w}_1 + w_1^2 + 2w_0w_2 = 0, \quad \text{........................(7c)}$$

$$\dots\dots\dots\dots\dots\dots\dots.$$

From (7a), $\quad w_0 = \pm i\rho. \quad$ .........................................................(8)

From (7b), (8), $\quad w_1 = -\tfrac{1}{2}\dot{w}_0/w_0,$

so

$$\int^t w_1\, du = -\tfrac{1}{2}\int^t dw_0/w_0 + \text{constant}$$

$$= \log A\rho^{-1/2} \mp \tfrac{1}{4}\pi i. \quad \text{.......................................(9)}$$

* *Annalen der Physik*, **19**, 585, 1934.

† Combining the two substitutions, we get

$$y = \exp\{\nu \int^t (w_0 + w_1/\nu + w_2/\nu^2 + \dots)\, du\}.$$

The substitution $\quad y = e^{\nu w}\phi(1 + f_1/\nu + f_2/\nu^2 + \dots)$

was made by H. Jeffreys on obtaining approximate solutions of Mathieu's equation, $d^2y/dt^2 + (a - 2k^2 \cos 2t)y = 0$. (*Proc. L.M.S.*, **23**, 429, 1923.) The first substitution gives a closer approximate solution than the second, and is easier to use.

From (7c),
$$w_2 = -(\dot w_1 + w_1^2)/2w_0$$
$$= (2w_0\ddot w_0 - 3\dot w_0^2)/8w_0^3$$
$$= \mp i\left(\frac{2\rho\ddot\rho - 3\dot\rho^2}{8\rho^3}\right). \quad\text{.......................................(10)}$$

Therefore
$$\nu\int^t w\,du = \int^t \{\nu w_0 + w_1 + w_2/\nu + O(1/\nu^2)\}\,du$$
$$= \pm i\sqrt{a}\int^t \rho\,du + \log A\rho^{-1/2} \mp \tfrac{1}{4}\pi i \mp \frac{i}{8\sqrt{a}}\int^t \frac{2\rho\rho'' - 3\rho'^2}{\rho^3}\,du + O(1/a). \quad\text{......(11)}$$

Hence to a second approximation,
$$y_1(t),\ y_2(t) = \exp\left(\nu\int^t w\,du\right)$$
$$= B\rho^{-1/2}\exp\left\{\pm i\sqrt{a}\int_0^t \rho\,du \mp \frac{i}{8\sqrt{a}}\int_0^t \frac{2\rho\rho'' - 3\rho'^2}{\rho^3}\,du\right\}. \quad\text{................(12)}$$

Combining the two solutions in (12) with two arbitrary constants, we obtain Erdélyi's result, namely :
$$y(t) = K\{a - 2k^2\psi(\omega t)\}^{-1/4}\cos\chi(t), \quad\text{.......................(13a)}$$
where
$$\chi(t) = \int_0^t \{a - 2k^2\psi(\omega u)\}^{1/2}\,du + \frac{k^2\omega^2}{4a\sqrt{a}}\int_0^t \left\{\psi''(\omega u) + \frac{5k^2}{2a\rho^2}\psi'^2(\omega u)\right\}\frac{du}{\rho^3} + \theta_0, \quad\text{...(13b)}$$
$\theta_0$ being an arbitrary constant. Using Jeffrey's substitution, the second integral in (13*b*) is not obtained.

2. *Mathieu's equation.*

This is
$$d^2y/dt^2 + (a - 2k^2\cos 2t)y = 0,$$
so an approximate solution subject to the conditions given in § 1 is obtained by writing $\cos 2u$ for $\psi(\omega u)$ in § 1 (13). If the second integral and what follows is omitted, we get Jeffrey's result, namely :
$$y(t) = K\{a - 2k^2\cos 2t\}^{-1/4}\cos\{(a + 2k^2)^{1/2}E(\lambda, t)\}, \quad\text{...............(1)}$$
where $E$ is an incomplete elliptic integral of the second kind with modulus $\lambda = 2k/\sqrt{(a + 2k^2)} < 1$, that is,
$$E(\lambda, t) = \int_0^t \sqrt{(1 - \lambda^2\cos^2 u)}\,du.$$

3. *Type of function represented by* § 2, (1).

The function is oscillatory, having an envelope dependent upon the factor $(a - 2k^2\cos 2t)^{1/4}$, and $y(t)$ fluctuates in amplitude between the approximate limits $(a \pm 2k^2)^{-1/4}$. By virtue of the argument $(a + 2k^2)^{1/2}E(\lambda, t)$, the rate of oscillation or " instantaneous frequency " fluctuates also. It is defined to be
$$\frac{1}{2\pi}\frac{d}{dt}\{(a + 2k^2)^{1/2}E(\lambda, t)\} = \frac{1}{2\pi}(a + 2k^2)^{1/2}(1 - \lambda^2\cos^2 t)^{1/2}. \quad\text{............(1)}$$
This has minima at $t = n\pi$ and maxima at $t = (n + \frac{1}{2})\pi$.

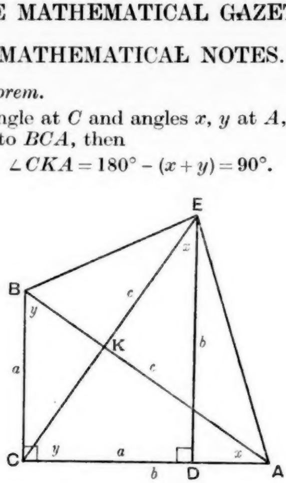
In the language of radio technology, this function exhibits both amplitude and frequency modulation (fluctuation). Either one or the other is necessary in order to convey signals in radio transmission (broadcasting). Moreover, the analysis is of more than academic interest. N. W. McL.

## MATHEMATICAL NOTES.

**1813.** *Pythagoras' theorem.*

If $ABC$ has a right-angle at $C$ and angles $x$, $y$ at $A$, $B$, and if $CDE$, placed as shown, is congruent to $BCA$, then

$$\angle CKA = 180° - (x+y) = 90°.$$




Then area $ACBE$ = area $CBE$ + area $ACE$,

or
$$\tfrac{1}{2}c^2 = \tfrac{1}{2}a^2 + \tfrac{1}{2}b^2.$$

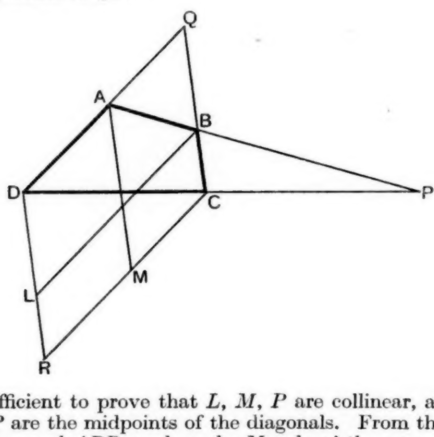
This is probably the shortest and simplest proof based on areas.

R. Hamilton Dick.

**1814.** *The complete quadrilateral.*

The following is a very simple and short proof of the fact that the mid-points of the diagonals of a complete quadrilateral are collinear.

Let $ABCD$ be the quadrilateral, and let $AB$, $DC$ meet at $P$, $AD$, $CB$ meet at $Q$. Complete the parallelogram $QDRC$, and draw $BL$, $CR$ parallel to $QD$ and $AM$, $DR$ parallel to $QC$.




It will be sufficient to prove that $L$, $M$, $P$ are collinear, as the midpoints of $QL$, $QM$, $QP$ are the midpoints of the diagonals. From the triangle $QDC$, taking the transversal $ABP$, we have by Menelaus' theorem

$$QA \,.\, DP \,.\, CB = -AD \,.\, PC \,.\, BQ\,;$$

hence by taking the opposite sides of parallelograms we have

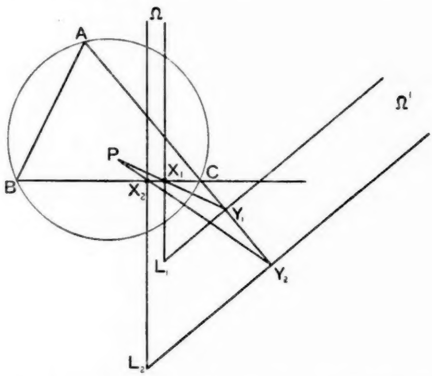
$$CM \,.\, DP \,.\, RL = -MR \,.\, PC \,.\, LD,$$

and hence $L$, $M$, $P$ are collinear by the converse of Menelaus' theorem for the triangle $RDC$.

It will be remembered that there was an old method, by completing the parallelograms $QDLB$, $QAMC$ and showing that $L$, $M$, $P$ were collinear by similar triangles. The above method is simpler. H. N. HASKELL.

1815. *The pedal lines of a given point.*

To prove that there are in general three pedal lines passing through a given point $P$. (The following proof was given by Lear P. Wood, a VIth Form boy at Harrison College, and Barbados Scholar for 1942. He is now at McGill University.)




Let any line through $P$ cut $BC$ and $AC$ at $X_1$ and $Y_1$. Draw perpendiculars at $X_1$ and $Y_1$ to $BC$ and $AC$ meeting in $L_1$. Similarly for $L_2$, $L_3$, ... .

Now $$(X_1X_2 \ldots) = (Y_1Y_2 \ldots);$$
hence $$\Omega(X_1X_2 \ldots) = \Omega'(Y_1Y_2 \ldots).$$

Thus the locus of $L$ is a hyperbola, which, in general, cuts the circle in four points, of which one is the point $C$. The pedal lines of the other three points of intersection will pass through $P$.

The reason why the pedal line of $C$ does not pass through $P$ is that the join of $C$ to $P$ does not necessarily pass through the foot of the perpendicular from $C$ to $AB$; while on the other hand, the join of $X_r$ to $Y_r$ does pass through the foot of the perpendicular from $L_r$ to $AB$, where $L_r$ is on the circle.

L. P. WOOD. (Per H. N. HASKELL.).

1816. *Mental multiplication.*

The following very simple tricks for mental multiplication do not seem to be widely known.

1. If $b + c = 10$, $(10a + b)(10a + c) = 100a(a + 1) + bc$.

Examples :

| | | |
|---|---|---|
| $84 \times 86 = 7224$. | Calculation : $8 \times 9 = 72$, | $4 \times 6 = 24$. |
| $113 \times 117 = 13221$. | $11 \times 12 = 132$, | $3 \times 7 = 21$. |
| $125^2 = 15625$. | $12 \times 13 = 156$, | $5^2 = 25$. |

2. If $a+c=10$, $(10a+b)(10c+b)=100(ac+b)+b^2$.

Examples :

$47\times 67=3149$. Calculation : $4\times 6+7=31$, $7^2=49$.

$23\times 83=1909$. $2\times 8+3=19$, $3^2=\ 9$.

2·1. If $a+c=100$, $(100a+b)(100c+b)=10^4(ac+b)+b^2$.

Example : $4017\times 6017=24170289$.

3·1. If $a+b$ is even, $(10a+5)(10b+5)=100\{ab+\frac{1}{2}(a+b)\}+25$.

3·2. If $a+b$ is odd, $(10a+5)(10b+5)=100\{ab+\frac{1}{2}(a+b-1)\}+75$.

Examples :

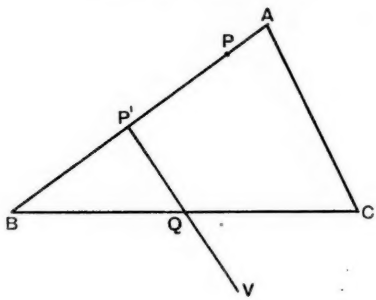
$115\times 165=\ 18975$. Calculation : $11\times 16+\frac{1}{2}(11+16-1)=189$.

$835\times 875=730625$. $83\times 87+\frac{1}{2}(83+87)=7221+85$.

R. L. Goodstein.

**1817.** *On Note* 1730 (*Gazette*, XXVIII, July, 1944).

The following construction for a line through a given point to bisect a given triangle is perhaps longer both to describe and to perform than those given in Note 1730, but is surely easier to understand.




For any point $P$ on $AB$, take $Q$ on $BC$ so that

$$\triangle PBQ=\tfrac{1}{2}\triangle ABC,$$

that is, draw $BQ$ equal to $AB\,.\,BC/2BP$. Join $VQ$ to cut $AB$ at $P'$. The range described by $P$ is projective to that described by $Q$, because of the algebraic relation $BQ=k/BP$ ; and that range is projective to the range described by $P'$, since they are both sections of the same pencil. Hence to solve the problem we have only to find the double point or points of the ranges $P$, $P'$ which lie in the segment $AB$. F. G. Maunsell.

**1818.** *On Note* 1717 ("*Inaccessible heights and distances*").

I suppose that most of us teach our rudiments of trigonometry in the order tangent, sine, cosine ; then, after we (as well as our pupils) have had a surfeit of ladders leaning against walls, guyropes attached to flagpoles, and the like, we feel the need for something more stimulating to serve the dual purpose of extending the work already done and of taking down a peg the cocky youngster who thinks it all too easy! Sines and cosines are well served by problems on northing and easting—or, what is their equivalent, the composition of forces at a point—but tangents get very little look-in here, their only appearance being at the very end when the direction of the resultant course—or force—

is to be determined. It is here that the type of problem indicated by Dr. Langford is so useful, though I would suggest a method of solution rather different from that given by him.

The problem, usually known as "Inaccessible Heights and Distances", commonly arises in any of three different forms :

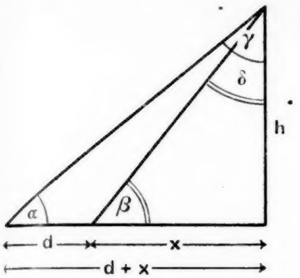


FIG. 1.

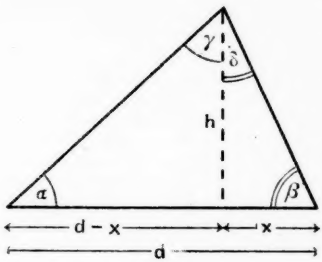


FIG. 2.

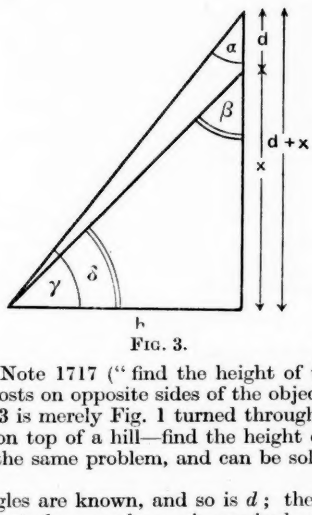


FIG. 3.

Fig. 1 is as given in Note 1717 ("find the height of the hilltop"); Fig. 2 has the observation posts on opposite sides of the object ("find the height of the balloon"); Fig. 3 is merely Fig. 1 turned through a right angle ("flag-staff of given height on top of a hill—find the height of the hill"). But all three are essentially the same problem, and can be solved *very easily* by the same special rule.

In all cases the angles are known, and so is $d$; there are two unknowns, $h$ and $x$, of which almost always only one is required. The rule to drill home is this :

**Always use the Angles opposite the Length that is <u>not</u> required.**

Thus, if we require $h$, we use angles $\gamma$ and $\delta$ (opposite to $x$), and the solution becomes :

$$x = h \tan \delta \; ; \quad \left.\begin{matrix} d+x \\ d-x \end{matrix}\right\} = h \tan \gamma .$$

Add or subtract, as the case may be, and $h$ follows at once—the elimination of $x$ is automatic.

In the case of Fig. 3 we should probably be asked for $x$, and so we should use angles $\alpha$ and $\beta$ (opposite to $h$). In this case, we have :

$$x \tan \beta = h = (d + x) \tan \alpha\text{—whence } x \text{ immediately.}$$

If this rule is employed, there is no longer any question of the problems being "sufficiently awkward to put quite a number of pupils off in despair", and, in my opinion, they are worthy of a place in our courses before we get to logarithms and the sine rule. G. H. GRATTAN-GUINNESS.

**1819.** *Some properties of plane cubic curves.*

The following properties of plane cubic curves seem to be unfamiliar. The general theory of such curves is so well known that it will be sufficient to state the theorems with a very brief indication of their proofs.

Through the node $O$ of a nodal cubic and any point $P$ of its plane four conics pass, each having four-point contact with the cubic. If their points of contact subtend at $O$ a pencil of constant cross-ratio, the locus of $P$ is a cubic having as inflexions and inflexional tangents the inflexions and inflexional tangents of the given curve. If the conic through $O$ and the four points of contact goes through another fixed point, the locus of $P$ is a conic through $O$.

If $O$ is an acnode (isolated point), the cubic may be projected into the symmetrical curve $r \sin 3\theta = 3a$. Let $ABC$ be the triangle formed by the inflexional tangents (the asymptotes). Then the conics of four-point contact are all real, if $P$ lies inside the triangle ; and none are real, if $P$ lies between the curve and two sides of the triangle produced. Otherwise two of the conics are real. The conics, if real, are hyperbolas.

There are four conics through $A$, $B$, $C$ having double contact with the conic through $O$, which has four-point contact at $Q$. As $Q$ varies, the chords of contact envelop conics touching $BC$, $CA$, $AB$.

If $O$ is a crunode, the cubic may be projected into the Folium $x^3 + y^3 = 3axy$. None of the conics are real, if $P$ lies inside the loop or between the curve and the asymptote. Otherwise two are real.

Similar results can be obtained for the cuspidal cubic.

To prove these properties we may take the cubic as the locus of

$$(t^3 + at,\ b,\ t^2 + a),$$

the conic through $O(0, 1, 0)$ and the points whose parameters are the roots of

$$t^4 - s_1 t^3 + s_2 t^2 - s_3 t + s_4 = 0$$

being $$bx^2 - (s_3 - as_1)xy - bs_1 zx + (s_4 - as_2 + a^2)zy + (s_2 - a)bz^2 = 0.$$

If $f(x, y, z) = 0$ is a plane algebraic curve,

$$x\frac{\partial f}{\partial X} + y\frac{\partial f}{\partial Y} + z\frac{\partial f}{\partial Z} = 0$$

is the "polar line" of $(X, Y, Z)$, which is one of its "poles". The number of poles of a line for a cubic is in general four, or three for a nodal cubic and two for a cuspidal.

Let $A$, $B$, $C$, $O$ be four real points. Let $AO$, $BO$, $CO$ meet $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$ and meet a line $l$ at $A_2$, $B_2$, $C_2$.

There is one cubic with $O$ as node having $A$, $B$, $C$ as the poles of $l$. The node is an acnode, if three or one of the ranges $(A_1A_2OA)$, $(B_1B_2OB)$, $(C_1C_2OC)$ have their cross-ratios positive. If two or none of the cross-ratios are positive, the node is a crunode.

If the node and the three poles of a line $l$ are given and also one other point on the cubic, the envelope of $l$ is a class-cubic.

This follows from the fact that, if $A$, $B$, $C$, $O$ are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 1)$ and the line $l$ is $\lambda x + \mu y + \nu z = 0$, then the cubic is

$$(\mu+\nu)(2\lambda+\mu+\nu)x^2(\lambda x + 3\mu y + 3\nu z) + \ldots + \ldots = 6\{(\lambda+\mu+\nu)^3 + \lambda\mu\nu\}xyz.$$

If $C$ is real but $A$ and $B$ are conjugate complex, then $O$ is an acnode or crunode according as the range $(C_1C_2OC)$ has a negative or positive cross-ratio.

If we have four points and a line $l$ given, there is a pencil of non-singular cubics, all meeting $l$ at the same three points, having three-point contact with each other there, and having the four given points as the poles of $l$.

Taking $l$ as $x+y+z=0$ and these poles as $(\lambda, \pm\mu, \pm\nu)$, where $\lambda > \mu > \nu > 0$, the cubics are

$$k\left(\frac{x^3}{\lambda^2}+\frac{y^3}{\mu^2}+\frac{z^3}{\nu^2}\right) = (x+y+z)^3,$$

$k$ being a variable parameter.

If $k$ lies between $(\lambda+\mu+\nu)^2$ and $(\lambda+\mu-\nu)^2$ or between $(\lambda-\mu+\nu)^2$ and $(-\lambda+\mu+\nu)^2$, the cubic has two circuits. One or two of these two-circuited cubics are harmonic according as $\lambda < \mu+\nu$ or $\lambda > \mu+\nu$. For other values of $k$ the cubic is one-circuited. This follows from the fact that with the notation of Salmon's *Higher Plane Curves*, §§ 217–221, $T^2 + 64S^3$ is negative or positive according as the cubic has two circuits or one.

If the four given poles form two conjugate complex pairs, all the cubics are one-circuited.

If two of them are real and two conjugate complex, we may take their coordinates as $(\pm\lambda, 0, 1)$, $(0, \pm\mu i, 1)$, where $\lambda$ and $\mu$ are real and positive. The cubics are

$$k\left(\frac{x^3}{\lambda^2}+\frac{3x^2y}{\lambda^2}-\frac{3xy^2}{\mu^2}-\frac{y^3}{\mu^2}+z^3\right) = (x+y+z)^3.$$

The cubic is two-circuited, if $k$ lies between $(\lambda+1)^2$ and $(\lambda-1)^2$. One or two of these two-circuited cubics are harmonic according as $\lambda > 1$ or $\lambda < 1$.

Suppose that $ABC$ is any given triangle and $l$, $m$ are any given lines. Let $L$ be the pole of $l$ for the triangle, so that $L$ is $(1/\lambda, 1/\mu, 1/\nu)$, if $ABC$ is the triangle of reference and $l$ is $\lambda x + \mu y + \nu z = 0$. Then the poles of $l$ for all cubics, with the sides of the triangle as real inflexional tangents and $m$ as the line of real inflexions, lie on the conic through $A$, $B$, $C$, $L$ and the intersection of the harmonic polars of the real inflexions. The conic meets both $l$ and $m$ in real points. (See Hilton's *Plane Algebraic Curves*, Ch. XIV, §§ 6–9.)

For a family of syzygetic cubics with the same nine inflexions, there are three real lines each containing one real and two unreal inflexions. Let them form a triangle $ABC$ and let $L$ be the pole of a given line $l$ for the triangle $ABC$. Then the poles of $l$ for all the cubics lie on the quartic, having inflexions at the twelve critic centres of the cubics and touching $LA$, $LB$, $LC$ at $A$, $B$, $C$.

The conic through $A$ and the poles divides $BC$ harmonically and touches $LA$.

H. Simpson.

1820. *A note on a cubic and an associated family of conics.*

From any point $P$ on a non-singular cubic curve it is possible to draw four tangents to the curve. Let the points of contact be $A$, $B$, $C$, and $D$, and through any three of them, $A$, $B$, $C$, and the point $P$ draw a conic to cut the cubic again in the points $L$ and $M$. It is then well known that the join of $L$ and $M$ always passes through the fourth point of contact $D$.* It is proposed

* Salmon, *Courbes Planes*, p. 190 and p. 193.

to show that the tangents to the conic and cubic at the point $L$ and the two lines $LP$ and $LD$ form a harmonic pencil, and hence that the tangents to the cubic at $L$ and $M$ intersect on the conic.

Taking $L(1, 0, 0)$, $P(0, 1, 0)$ and $D(0, 0, 1)$ as triangle of reference, the equation of the cubic may be expressed as

$$z^2x + z(ax^2 + 2hxy + by^2) + xy(fx + gy) = 0.$$

The polar conic of $P$ is then

$$2hxz + 2byz + fx^2 + 2gxy = 0.$$

Since this polar conic passes through the points $A$, $B$, $C$, $D$ and touches the cubic at $P$, the nine points $A$, $B$, $C$, $D$ (twice), $L$, $M$, and $P$ (twice) may be taken as the base points of a pencil of cubics. Hence eliminating the terms without a factor $x$ and taking out this factor from the result, the equation of the conic $ABCLMP$ is

$$2z^2 + 2z(ax + hy) + fxy = 0.$$

The tangents at $L$ to the conic and cubic are therefore

$$2az + fy = 0, \quad \text{and} \quad az + fy = 0,$$

and they are such that the line $LD$ and the conic tangent harmonically separate the line $LP$ and the cubic tangent. The same property holds for the similar pencil at $M$.

Let the tangents to the cubic at $L$ and $M$ cut the conic again in $T$ and $T'$.

Then $\qquad L(LPMT) = P(LPMT) = -1,$

and $\qquad M(LPMT') = P(LPMT') = -1,$

and the points $T$ and $T'$ coincide; that is, the tangents at $L$ and $M$ to the cubic meet on the conic.

The locus of this point of intersection $T$ for varying conics through the four points $P$, $A$, $B$, $C$, may be obtained by taking $ABC$ as triangle of reference and choosing suitable scales so that $P$ is the unit point. The cubic then has an equation of the form

$$ax^2(y - z) + by^2(z - x) + cz^2(x - y) = 0,$$

and the coordinates of $D$ are $(bc, ca, ab)$. Taking any line through $D$ as $DLM$ its equation is

$$ax - bz + t(by - cz) = 0,$$

and the combined equations of the lines $AL$, $AM$ is therefore

$$bt(1 + t)y^2 - (c - a + 2ct + bt^2 + ct^2)yz + ct(1 + t)z^2 = 0.$$

The harmonic conjugate $AT$ of $AP$ with respect to $AL$, $BM$ is then

$$(b - c)(y - z)t^2 + 2(b - c)yt - (c - a)(y + z) = 0.$$

Similarly the equation of $BT$ is

$$(b - c)(z + x)t^2 - 2(c - a)xt + (c - a)(z - x) = 0.$$

Eliminating $t$, the equation of the locus of $T$ may be simplified to

$$(b - c)y^2z^2 + (c - a)z^2x^2 + (a - b)x^2y^2 = 0,$$

a quartic having "biflecnodes" at $A$, $B$, and $C$, and passing through $P$.

In the case of a nodal cubic only two tangents can be drawn from $P$, touching at $A$ and $D$. If $O$ is the node, and $DLM$ any line through $D$, then the conic through $P$, $L$, $M$, $A$ and $O$ gives the same harmonic property at $L$ and

$M$. If the tangents from $D$ to the cubic touch at $R$ and $S$, then $RS$ passes through $A$, and the conic $PLMAO$ touches at $O$ the harmonic conjugate $OQ$ of $OA$ with respect to $OR$, $OS$. The locus of $T$ is a cubic with a cusp at $O$, tangent $OQ$, and passing through $A$ and $P$. L. E. Prior.

**1821.** *On "bending momentum round corners".*

From a long experience of teaching elementary mechanics to university students who have passed the Higher School Certificate Examination, I find that ideas about impulse and momentum are often vague and sometimes erroneous. The methods used in answering questions involving these concepts in various Higher School examination papers lead me to conclude that some teachers find the teaching of the relation between impulse and momentum difficult.

Since impulse can only be mathematically defined as an integral, some knowledge of the integral calculus is necessary before the concept can be satisfactorily introduced. For this reason elementary treatments frequently restrict their attention to problems in which the force is constant and the change of momentum takes place in one straight line.

This tends to obscure the fundamental fact that impulse, and the change of momentum by which it is measured, are vectors.

The vector equation of motion of a particle moving in a plane is

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F},$$

and, on integrating this from $t=0$ to $t=\tau$, we get

$$m\mathbf{v}_1-m\mathbf{v}_0=\int_0^\tau \mathbf{F}\,dt, \qquad\qquad (1)$$

where $\mathbf{v}_0$ is the velocity at time $t=0$ and $\mathbf{v}_1$ is that at time $t=\tau$. The integral on the right is defined to be the *impulse* of the force in the period of time $\tau$. Alternatively, we may start from the differential standpoint and write

$$\Delta(m\mathbf{v})=\mathbf{F}\,\Delta t,$$

and integration leads to equation (1).

The integral on the right of (1) is the limit of a sum of the same type as occurs in the elementary definition of a definite integral. A geometrical interpretation can be given of the integral as the area under the force-time curve between the ordinates at $t=0$ and $t=\tau$.

Since $\mathbf{F}$ is a vector function of $t$, the integral is also a vector function of the time $\tau$.

When we wish to apply this relation between impulse and momentum to estimate the effect of a sudden blow administered to the particle at time $t=0$, we define the *instantaneous impulse* as the limit of the above integral as $\tau\to 0$. We assume that this limit exists by supposing that the force acting on the particle becomes very great during a very short interval of time. Denoting this limit, assumed finite, by the vector $\mathbf{J}$, we have the equation

$$m\mathbf{v}_1-m\mathbf{v}_0=\mathbf{J}, \qquad\qquad (2)$$

where $\mathbf{v}_0$ is the velocity just before and $\mathbf{v}_1$ the velocity just after the blow has been administered.

Since impulse has dimensions $MLT^{-1}$, it is of different dimensions from force, and so it is better not to use the term "impulsive force", which is used in some textbooks, but retain the term "impulse" only for the vector $\mathbf{J}$ of equation (2).

F

Many of the problems to illustrate this principle are concerned with two particles joined by a taut inelastic string. If an impulse is applied to one of the particles in an assigned direction, an impulsive stress is, in general, set up in the string. It is sometimes convenient to speak of this as an " impulsive tension ", but if this term is used it is important to emphasise that it is not the same as the ordinary tension in the string, but is an impulse which only acts at the instant at which the external impulse is applied to the system.

Failure to emphasise the fundamentally vectorial character of impulse leads to an erroneous method of solving elementary problems which may be conveniently described by the phrase (first coined, I believe, by my friend and colleague Mr. A. C. Stevenson) " bending momentum round corners ".

This is best illustrated by considering two elementary problems of a type commonly set in examination papers.

*Example* 1. If two particles $A$, $B$ of masses $m_1$, $m_2$ lie on a smooth horizontal table and are connected by an inelastic string, and $B$ is projected along the line joining the particles with velocity $u$ until the string tightens, at the instant at which the string becomes taut an impulse (impulsive tension) is instantaneously set up in the string. Denote this by **J**. If after the string tightens both $A$ and $B$ begin to move with velocity $v$, the equations of impulsive motion are :


For $A$ : $\quad m_1 v - m_1 0 = J.$

For $B$ : $\quad m_2 v - m_2 u = -J.$

On adding these we get,

$$(m_1 + m_2) v - m_2 u = 0,$$

which is equivalent to the statement that ***the total momentum along the line AB of the system A + B before the impulse is applied is equal to the total momentum of A + B along AB after the impulse.*** In consequence of this, the latter statement tends to be taken as the fundamental principle. This does little harm so long as the motion is all *in the same straight line.*

The error creeps in if the " total momentum " idea is carried over to the similar problem in which the string passes over a smooth peg or small pulley, as in Attwood's machine.

*Example* 2. Two masses $A$, $B$ of $3m$ lb. and $2m$ lb. are held in contact, but are connected by an inelastic string 5 ft. long which hangs loosely. From $A$ another inelastic string passes over a smooth peg vertically above $A$ and supports a mass $C$, of $5m$ lb., hanging freely at the other end. The masses $A$ and $B$ are simultaneously released. Show that, when the string $AB$ tautens, the system is brought to instantaneous rest.

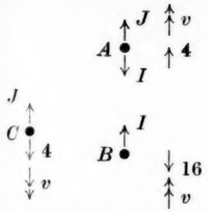
Before the string $AB$ tautens, let $C$ be descending with acceleration $f$. Since the peg is smooth the tension $T$ in the string $AC$ is the same throughout its length. Hence the equations of motion of $C$ and $A$ are :

$$5mf = 5mg - T, \qquad 3mf = T - 3mg,$$

and, on eliminating $T$, we get $f = g/4 = 8$ ft./sec.$^2$

The relative acceleration of $A$ and $B$ is $32 + 8 = 40$ ft./sec.$^2$, and since the string $AB$ is 5 ft. long, we readily find that the time until $AB$ tautens is $\frac{1}{2}$ sec., and, at the instant that the string $AB$ becomes taut, the velocity of $A$ is 4 ft./sec. upwards and that of $B$ is 16 ft./sec. downwards.

Let the impulses in the strings $AB$, $AC$ be $I$, $J$, and suppose that, after the jerk due to the tightening of the string $AB$, the whole system has velocity $v$ ft./sec. For clarity it is best to draw diagrams of each mass separately. From the figures, we see at once that the equations of impulsive motion are :




For $C$ : $\quad -J=5m(v-4)$.

For $A$ : $\quad J-I=3m(v-4)$.

For $B$ : $\quad I=2m\{v-(-16)\}$.

On adding these we readily see that $v=0$, and the impulses $I$ and $J$ may be calculated, if required.

This solution is strictly in accordance with the principles of impulsive motion. To show that $v=0$ it is often erroneously argued that the "total momentum of the system $A$, $B$, $C$ is unchanged by the impulse", but it is clear, when we remember that impulse and momentum are vectors, that there is no such thing as the "total momentum of the system $A$, $B$, $C$"; in other words, "momentum cannot be bent round corners". E. G. Phillips.

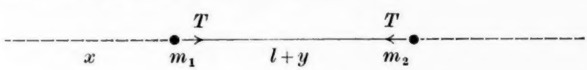
1822. *A point in teaching dynamics.*

*Question.* Two particles $m_1$, $m_2$ connected by a light inelastic string rest on a smooth horizontal plane. The string is slack and $m_2$ is projected with velocity $u$ away from $m_1$. What are the velocities when both particles are in motion?

*Solution.* Let $v$ be the common velocity after $m_1$ is jerked into motion; by the conservation of momentum, $(m_1+m_2)v=m_2u$.

*Query.* Is it possible to justify the assumption above that $m_1$ and $m_2$ will have a common velocity? Most boys, when asked to think about this, say that $m_1$ will have a greater velocity than $m_2$.

*Answer.* In practice the string must be slightly elastic; $m_1$ cannot instantaneously acquire a velocity $v$. Let $l$ be the natural length of the string and $\lambda$ its (very large) modulus of elasticity.




Then $$T=m_1\ddot{x}=-m_2(\ddot{x}+\ddot{y})=\lambda y/l.$$

Eliminating $\ddot{x}$, we have $\ddot{y}=-\mu y$, where $\mu=\lambda(m_1+m_2)/lm_1m_2$.

The initial conditions are : $\quad t=0, \quad x=0, \quad y=0, \quad \dot{x}=0, \quad \dot{y}=u.$

Thus $$\dot{y}=u\cos t\sqrt{\mu} \quad \text{and} \quad y=(u/\sqrt{\mu})\sin t\sqrt{\mu},$$

whence $$\ddot{x}=(\lambda u/m_1l\sqrt{\mu})\sin t\sqrt{\mu},$$

and $$\dot{x}=\lambda u(1-\cos t\sqrt{\mu})/m_1l\mu=v(1-\cos t\sqrt{\mu}).$$

The velocity of $m_1$ is $v-v\cos t\sqrt{\mu}$,

the velocity of $m_2$ is $v+(u-v)\cos t\sqrt{\mu}$,

and the period $2\pi/\sqrt{\mu}$ is very small.

| *Time.* | *Velocity of $m_1$.* | *Velocity of $m_2$.* | *String.* |
|---|---|---|---|
| 0 | 0 | $u$ | natural length ; |
| $\pi/2\sqrt{\mu}$ | $v$ | $v$ | greatest extension ; |
| $\pi/\sqrt{\mu}$ | $2v$ | $2v-u$ | natural length. |

When the extension of the string is greatest, the masses have the common velocity $v$, and this state of affairs will continue if the string is permanently deformed. In this case the assumption is justified.

If the string is elastic, when the velocities are $2v$, $2v-u$ and the string is at its natural length, it will go slack and the simple harmonic motion ceases, so that those velocities are maintained, thus confirming the boys' guesses. Note that $2v-u=(m_2-m_1)u/(m_1+m_2)$, and so $m_2$ will be jerked backwards if $m_1>m_2$.

In practice the actual behaviour will probably be somewhere between the above two cases. M. A. PORTER.

**1823.** *Curiosités arithmétiques.*

*A.* Voici les couples de nombres de même parité consécutifs dont le produit contient les chiffres 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, pris une fois.

$$35577 \times 35579 = 1265794083, \qquad 63727 \times 63729 = 4061257983,$$
$$41830 \times 41832 = 1749832560, \qquad 76624 \times 76626 = 5871390624,$$
$$46836 \times 46838 = 2193704568, \qquad 78426 \times 78428 = 6150794328,$$
$$49957 \times 49959 = 2495801763, \qquad 83493 \times 83495 = 6971248035,$$
$$53773 \times 53775 = 2891643075, \qquad 85554 \times 85556 = 7319658024,$$
$$97776 \times 97778 = 9560341728.$$

Les produits suivants sont de formes comparables :

(*a*) $54135 \times 54139 = 2930814765,$
$59553 \times 59557 = 3456798021,$
$60372 \times 60376 = 3645019872.$

(*b*) $35910 \times 35916 = 1289743560,$
$62124 \times 62130 = 3859764120,$
$62880 \times 62886 = 3954271680,$
$63054 \times 63060 = 3976185240.$

(*c*) $51147 \times 51167 = 2617038549,$
$51892 \times 51912 = 2693817504,$
$54198 \times 54218 = 2938507164.$

(*d*) $43074 \times 43104 = 1857092433.$

*B.* Il n'existe aucun nombre entier dont le cube s'écrire avec les chiffres 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, pris une fois.* Mais, parmi les cubes de 11 et 12 chiffres, il y en a deux dont les derniers chiffres à droite forment une des permutations des dix chiffres significatifs :

$$2326^3 = 12584301976, \qquad 7616^3 = 441754320896.$$

Les cubes parfaits

$$3024^3 = 27653197824, \qquad 5032^3 = 127415392768,$$
$$9463^3 = 847396215847,$$

se terminent par des nombres formant une des permutations de neuf chiffres significatifs, zéro excepté. V. THÉBAULT.

**1824.** *Sur une généralisation du théorème de l'orthopôle.*

1. M. R. Goormaghtigh a établi par le calcul cette propriété qui généralise le théorème de l'orthopôle d'une droite, par rapport à un triangle : † $A_1$, $B_1$, $C_1$

* V. Thébault, *Mathesis*, 1936, p. 33.

† *American Mathematical Monthly*, 1929, p. 422.

*étant les projections orthogonales des sommets $A$, $B$, $C$ d'un triangle sur une droite $\Delta$ de son plan, $A_1'$, $B_1'$, $C_1'$ les points divisant $AA_1$, $BB_1$, $CC_1$ dans le même rapport*

$$A_1A_1' : A_1A = B_1B_1' : B_1B = C_1C_1' : C_1C = k,$$

*les perpendiculaires de $A_1'$, $B_1'$, $C_1'$ sur $BC$, $CA$, $AB$ respectivement, sont concourantes.*

Voici une démonstration fort simple qui précise la position de l'orthopôle généralisé.

Les perpendiculaires de $A_1$, $B_1$, $C_1$ sur $BC$, $CA$, $AB$ concourent en l'orthopôle $M_1$ de $\Delta$, par rapport au triangle $ABC$, et les perpendiculaires de $A$, $B$, $C$ sur $BC$, $CA$, $AB$ concourent en l'orthocentre $H$ du triangle $ABC$. Les perpendiculaires de $A_1'$, $B_1'$, $C_1'$ sur $BC$, $CA$, $AB$ concourent donc en un point $M_2$ de $M_1H$ divisant ce segment rectiligne dans le rapport $M_1M_2 : M_1H = k$ (orthopôle généralisé). Les triangles $A_1'B_1'C_1'$, $ABC$ sont donc, à la fois, orthologiques et homologiques et la droite $M_2M_3$ qui joint les centres d'orthologie $M_2$, $M_3$ est perpendiculaire à $\Delta$ ; l'orthocentre $H_1'$ du triangle $A_1'B_1'C_1'$ partage le segment rectiligne $M_1'M_3$ joignant l'orthopôle $M_1'$ de $\Delta$ pour le triangle $A_1'B_1'C_1'$ au centre d'orthologie $M_3$, dans le rapport

$$M_1'H_1' : M_1'M_3 = k.$$

2. Les droites menées par $A_1$, $B_1$, $C_1$ coupant les côtés $BC$, $CA$, $AB$ sous un même angle $\theta$ déterminent un triangle $A_2B_2C_2$ semblable à $ABC$, le rapport de similitude étant $|\cos\theta|$. Le cercle $(\omega_1)$ circonscrit au triangle $A_2B_2C_2$ passe par l'orthopôle $M_1$ de $\Delta$. Lorsque $\theta$ varie, $\Delta$ restant fixe, le centre $\omega_2$ du cercle $(\omega_2)$ décrit le cercle $(\gamma)$ tracé sur $M_1O_2$ comme diamètre, $O_2$ étant le centre du cercle circonscrit au triangle $A_2B_2C_2$ symétriquement égal à $ABC$, dont les côtés passent respectivement par $A_1$, $B_1$, $C_1$. Il en résulte que *le cercle $(\omega_2)$ enveloppe une cardïoide, homothétique de la podaire de $(\gamma)$, $M_1$ étant le centre et 2 le rapport d'homothétie.*

Les droites menées par $A_1'$, $B_1'$, $C_1'$, coupant $BC$, $CA$, $AB$ sous le même angle $\theta$, forment un triangle $A_3B_3C_3$, semblable à $ABC$, le rapport de similitude étant $(1+k)\cos\theta$. Lorsque $k$ varie, $\theta$ restant constant, le centre $\omega_3$ du cercle $A_3B_3C_3$ décrit une droite indéfinie $\omega_1H$ sur laquelle se trouvent l'orthocentre $H$ de $ABC$, relatif à $k=1$, et son symétrique $H'$, par rapport à $\omega_1$, correspondant à $k=-1$. Le cercle $(\omega_3)$ enveloppe les tangentes au cercle circonscrit au triangle anticomplémentaire de $ABC$, issues du point $H'$.

Quand les points $A_1'$, $B_1'$, $C_1'$ sont fixes, $\theta$ étant variable, l'orthopôle généralisé $M_2$, tel que $\omega_1M_2 : \omega_1H = k$, est fixe, de même que le centre $O_4$ du cercle circonscrit au triangle $A_4B_4C_4$ homothétique à $ABC$ dont les côtés passent par $A_1'$, $B_1'$, $C_1'$ et tel que $O_2O_4 : O_2H = k$. Le lieu de $\omega_3$ est donc le cercle $(\Omega)$ décrit sur $O_2M_2$ comme diamètre. De plus, les cercles $(\omega_4)$ forment un faisceau du second ordre et sont donc orthogonaux à un cercle fixe $(\Gamma)$ qui passe en $M_2$. Il en résulte que les cercles $(\omega_3)$ enveloppent une anallagmatique du quatrième ordre dont la déférente est le cercle $(\Omega)$, c'est-à-dire une *spirique*.*

V. THÉBAULT.

**1825.** *Indefinite integration by means of contours.*

We all know how to find

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+x^2+1},$$

and therefore also

$$\int_0^{\infty} \frac{dx}{x^4+x^2+1},$$

* *Cfr.* H. Brocard et T. Lemoyne, *Courbes géométriques remarquables*, p. 4.

by proceeding to the limit from a semicircular contour. If we are equally familiar with the expression by means of residues of such integrals as

$$\int_c^\infty \frac{dx}{x^4+x^2+1} \qquad \text{.........................................(1)}$$

and

$$\int_c^\infty \frac{dx}{x^2-2x\cos\alpha+1}, \qquad \text{...................................(2)}$$

where the lower limit $c$ is an arbitrary real number, it is not our textbooks and treatises that we have to thank. Professor Hardy tells me that he used to give the evaluations forty years ago in lectures on elementary analysis, and he is as puzzled as I am at their failure to find their way into print in any of the obvious places.

(1) We may suppose $c>0$, and the integrand to be used is

$$\frac{1}{z^4+z^2+1}\log\frac{c+z}{c-z}.$$

The contour is a semicircle whose base is on the imaginary axis from $-iR$ to $iR$ and whose central radius, along the positive real axis, is slit from $c$ to $R$. Within this contour the logarithm is single-valued, its values on the two edges of the slit differ by $2\pi i$, and its value on the imaginary axis is imaginary. Hence the integral along the base of the semicircle is real, the contribution of the two edges of the slit is

$$2\pi i\int_c^R \frac{dx}{x^4+x^2+1},$$

and, proceeding to the limit, the value of the integral from $c$ to $\infty$ is twice the real part of the residue of the integrand at the pole in the first quadrant.

(2) The integrand is

$$\frac{\log(c-z)}{z^2-2z\cos\alpha+1},$$

and the contour is the complete circumference $|z|=R$ with the real axis slit as before from $c$ to $R$. The value of the required integral from $c$ to $\infty$ is now the negative of twice the real part of the residue of the integrand at the pole whose imaginary part is positive.

The second form of contour and integrand is the more fundamental, for the first form can be used only when the function concerned is even, and in that case the sum of the residues of $f(z)\log(c-z)$ at the four points $\pm a'\pm ia''$ is easily identified with the negative of the sum of the residues of $f(z)\log\{(c+z)/(c-z)\}$ at the two points $a'\pm ia''$. Professor Hardy points out that generality in the limit of the real integral is somewhat spurious, for

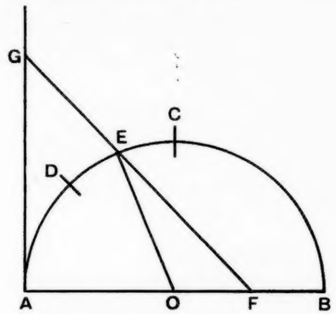
$$\int_c^\infty f(x)\,dx=\int_0^\infty f(c+x)\,dx,$$

and therefore if we have a process for dealing with an integral from 0 to $\infty$ the extension is trivial. This is true, but regarded as a function of $c$ the integral from $c$ to $\infty$ is the negative of an indefinite integral. In a particular integral such as that of $1/(x^2-2x\cos\alpha+1)$ from $\cos\alpha$ or 1 to $\infty$, a preliminary transformation is neither here nor there, but I am reluctant to obscure the significance of a process which determines an indefinite integral by means of a complete contour.

A fuller treatment of this subject, with a number of examples, has been sent to *The Mathematics Student* of Madras. E. H. N.

1826. *Approximate rectification of the circle.*

Is the following approximation well known? $OC$ is perpendicular to $AB$, $OD$ bisects the angle $AOC$, $OE$ bisects the angle $DOC$; $F$ is the mid-point of $OB$, and $FE$ meets the tangent at $A$ in $G$.




Then if $\angle EFO = \alpha$, $\angle OEF = 67\tfrac{1}{2}^\circ - \alpha$, and

$$\sin \alpha = 2 \sin (67\tfrac{1}{2}^\circ - \alpha),$$
$$\tan \alpha = 2 \sin 67\tfrac{1}{2}^\circ/(1 + 2 \cos 67\tfrac{1}{2}^\circ)$$
$$= 1{\cdot}0467, \text{ to five significant figures.}$$

Now $AG = \frac{3}{2}r \tan \alpha \simeq \frac{1}{2}\pi r$, the error being less than $\cdot 06\%$. B. A. SWINDEN.

## CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

SIR,—May a boy of the very Old Brigade mention that some of our successors seem to be appropriating laurels won by our age, and evidently forgotten? I take two instances from the *Gazette* for December, 1944, and if I do not quote the names of the contributors correctly, must ask their forgiveness, as my copy has gone to the Far East to cheer up a youthful mathematician somewhere in the jungle.

Mr. Hodgetts gives a proof of Feuerbach's Theorem which depends on a property of the limiting point of circles which touch. Now, in the 80's of the last century this proof was given in *Euclid Revised* by R. C. J. Nixon, who states it was given to him by Professor John Purser of Queen's College, Belfast. Mr. Nixon was a distinguished scholar of Peterhouse College, Cambridge: his *Euclid* was a great advance on its predecessors, replacing Todhunter's prolix version. In 1890 or thereabouts, I was a student in Dr. Purser's First Year, and in giving the proof he told us that his original proof was criticised by Mr. Nixon, and that he then fell back on the limiting point version.

In the note on Pythagoras' Theorem (due to Dr. Lawrence) I meet a method found in a still older *Euclid* than Nixon's—Galbraith and Haughton's, which enunciated all propositions in Greek as well as in English, but gave a valuable commentary after the proof. They gave the method of the note in the *Gazette* complete, and I often used it with students. In dealing with the discharged, wounded soldiers of 1918 I found they took to the construction of a parallelogram equal to the sum of two others, but boggled a bit over the niceties of the application to squares and a right-angled triangle.

Yours truly, F. W. HARVEY.

## REVIEWS.

**The Tyranny of Mathematics: an Essay in the Symbiosis of Science, Poetry and Religion.** By GEOFFREY HOYLAND. Pp. 52. 1s. 6d. 1945. (S.C.M. Press)

This is an important and timely little book: it deals with the responsibility of the scientist. The mind of the present age is turning more and more to science and the scientific approach for the solutions of its problems; and, as the author remarks in his preface, mankind is on the way to gain the whole material world and to lose its own soul in the process. Quite recently, in his presidential address to the Royal Society, Sir Henry Dale courageously challenged all scientists to face the future with full responsibility for their public part in the world of thought and action. There is indeed a danger that science will come under the sway of a false philosophy which will ultimately destroy science itself. And as a young scientist recently put it, once man was told how and why we live: now he is told how only.

As for the mathematician, who is so apt to wrap himself up in his beloved geometry and analysis, part of this responsibility is to apprehend and to realise what is in fact the resultant effect of his discoveries upon the world and the thought of the ordinary man. The author does us a service by making this duty very clear: but the book goes much further than this, for, as the preface states, it is an attempt to assess the causes and dangers of the present trend, and to suggest both the need for, and the possibility of, a true symbiosis of science and religion: that is, the true art of living together in mutual support.

There are three chapters. The first is a historical survey of the Newtonian Revolution which established the Tyranny of Mathematics. This tyranny began with the discovery of the calculus, and has infiltrated through all the physical and natural into the psychological sciences. Next follows a chapter on the limitations of the mathematical method, illustrated aptly enough by examples from art and poetry. Many a mathematician or scientist is of course well awhre of these limitations: but the private safeguards of the cultured few cannot be offset against the deadweight of an all-powerful and all-dominating influence of the scientific method as it impinges upon life to-day with a momentum acquired through three centuries of unhindered advance. Once this tyranny is recognised and the proper place of science in human existence is understood, what we need is some principle whereby we can judge all approaches to truth, including the mathematical. It is a great merit of the book that the author concludes his diagnosis with a general statement of five positive propositions: they form a solid contribution which bear sustained thinking out. They lead to a third chapter—Science and the Eternal—which develops the theme playfully, joyfully and seriously. The materialism of the eighteenth century, like the science of the nineteenth and twentieth, is in some ways an escape from reality, from the reality of the eternal. The shafts which the author directs are tipped with humour, but not with venom. He encourages us scientists to laugh at ourselves. One detects a new blossoming of the scientific spirit: the scientist needs a warm heart as well as a cool but calculating mind.

At a first reading I found the early pages provoking and then intriguing. There are details and a few typographical errors in the opening chapter which might distract the mathematically trained but unwary reader. It is a book for all who value mathematics to read, particularly for those of us who teach.

H. W. TURNBULL.

**From Atoms to Stars.** By M. Davidson. Pp. 188. 15s. 1944. (Hutchinson's Scientific and Technical Publications)

The conscientious reader of this book who starts it with a knowledge of physics and mathematics up to school-leaving standard can lay it down with a working acquaintance with the gist of modern astronomy acquired painlessly and—if he goes about the business intelligently—with a minimum of mental indigestion. More than this : the knowledge he has absorbed will be retained more easily than that acquired from most books on elementary astronomy, for the author has paid particular attention to underlying principles. His preface reads : " Numerous examples specially designed to illustrate points arising in the text have been inserted. . . ." It is true that the text is interlarded with numerical examples—nearly every page has at least one—fully worked out, but they give no impression of having been " inserted ". The author has woven them into the text skilfully, assuming the attitude of an enquiring arithmetician who likes to verify from first principles at least the order of magnitude of all the astronomical quantities he employs. The result is a book written in a most entertaining style, but one which can be appreciated to the full only by the reader who has pencil and paper and a slide rule beside him and is prepared to use them.

The book begins with an introductory chapter on the structure of the atom, which gets as far as neutrons and positrons in eight pages and the equivalence of mass and energy in twelve. Rather more leisurely chapters on the Sun, the Earth, the Moon, the planets, and comets and meteors follow ; these are mainly descriptive, and comprise about half the book. Much of the rest is taken up with the stars, the Galaxy, the external nebulae, and a sketch of the modern theory of stellar structure, all treated entertainingly enough, though on familiar lines. The penultimate chapter gives an excellent description of the current theory of energy generation in stellar interiors, which will be welcomed by those who have no easy access to the original papers. This section is well written, and the admittedly complex nuclear reactions are presented so clearly and simply that readers who have absorbed the introductory chapter will have no difficulty in understanding them. The author's treatment of the cosmogony of the solar system in his last chapter is perhaps less happy, and one wonders whether the subject should be more than touched upon in such a book as this.

It is possible to cavil at some of Dr. Davidson's physics—few will agree with his statement concerning the effect of telescopic aid on the twinkling of stars, for example, and the Sceptical Chymist would be embarrassed at the breadth of the law ascribed to him—and even at some of his arithmetic (" one radian is $\pi/180$ "), but apart from such minor slips, which can be corrected in the second edition which will no doubt soon be called for, the treatment is authoritative. That it is also up-to-date is attested by references to the most recent value for the solar parallax, to non-solar planets, and to the atmosphere of Titan. The author rightly loses no opportunity of directing attention to the valuable work of amateurs in astronomy, but this has tended to give a rather parochial air to parts of the book, and has led in one instance to distinct unfairness to the discoverer of Nova Puppis.

The standard of typography is adequate, but the illustrations have suffered badly from the imposition of " authorised economy standards ". The half-tone plates owe what impressiveness they retain to the fact that their originals are selected from the most spectacular photographs in existence. The many line diagrams are, however, good.

A popular work on a scientific subject which attains the distinction of a review in *Punch* (not in the book-reviewing columns, either!) must have much

to commend it to the lay reader. There is no doubt what this quality is: the book is clearly and entertainly written, and presents not merely the results of but also the methods used in modern astronomy in a way which can be followed by anyone with arithmetic and elementary physics at his call.

A. H.

**1. Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments.** Pp. xxxviii, 412. $7\frac{3}{4}'' \times 10\frac{1}{2}''$. 1943. 30s. (Columbia University Press, New York)

**2. Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.** Pp. (20), 408. $8'' \times 10\frac{3}{4}''$. 1940. 20s. (Federal Works Agency, W.P.A. for the City of New York)

**3. Tables of Sines and Cosines for Radian Arguments.** Pp. xx, 280. $8'' \times 10\frac{3}{4}''$. 1940. 20s. (Federal Works Agency, W.P.A. for the City of New York)

All the tables were prepared by the Mathematical Tables Project of the Work Projects Administration for the City of New York; technical director, Arnold N. Lowan. British Agents, Scientific Computing Service, 23 Bedford Square, London, W.C. 1.

The main object of the present review is to draw attention to the first volume listed above, but two others, amongst many which have not been noticed previously in the *Mathematical Gazette*, deal with closely related functions, so that it seems appropriate to describe them also, especially as they are of considerable importance. It is hoped that other volumes will be described later; these include one on the Exponential Function, four on Natural Logarithms, three on Sine, Cosine and Exponential Integrals, two on Probability Functions and one on the Inverse Tangent, as well as some minor tables.

The three volumes listed above together form the most extensive set of tables of circular and hyperbolic functions yet published, for which the argument is in radians. They, especially the third, should go a long way towards making this argument for the circular functions as easy to deal with numerically as it is analytically, in spite of the irrational period $2\pi$.

The close interval of tabulation is an important feature, as it allows use of linear interpolation to about 8 decimals throughout most of (1) and (2), which have interval 0·0001, and to 6 or 7 decimals in (3), which has interval 0·001; this should be ample for most users.

Descriptions of individual volumes follow:

(1) The main table gives values of $\tan x$, $\cot x$, $\tanh x$ and $\coth x$, to 8 or more figures (with 5 to 13 decimals) for $x=0(\cdot 0001)2$. The major part of this table is interpolable linearly, but obviously this is not possible throughout; second differences are given (sometimes for every argument on a page, sometimes for representative arguments only) whenever needed to obtain full accuracy, unless fourth differences are also needed, in which case none are given.

An auxiliary table gives 10-decimal values of all four functions for $x=0(\cdot 1)10$. There are also 15-decimal conversion tables between nonagesimal and radian measure, and tables of second-difference coefficients for Bessel's and Everett's interpolation formulae, for each 0·001 of an interval.

(2) The main table of this volume, the first of the three to be published, gives $\sin x$, $\cos x$, $\sinh x$, and $\cosh x$ to 9 decimals for $x=0(\cdot 0001)1{\cdot}9999$, with a supplementary table, also strictly to 9 decimals, for $x=0(\cdot 1)10$. No differences are given; interpolation is linear to nearly 9 figures throughout,

while if a second difference is needed, it may be found by omitting the final 8 digits of the entry used.

The most valuable feature of this table, as of the last, is the small interval of tabulation. It may also be worth noting that all four functions are given opposite a single argument; this is particularly useful for obtaining the circular and hyperbolic functions of argument $x\sqrt{(2i)} = (1+i)x$, that is, along the "semi-imaginary" axis.

(3) This important volume gives $\sin x$ and $\cos x$ to 8 decimals for $x = 0(\cdot 001)24\cdot 999$, with supplementary tables for $x = 0(1)100$ to 8 decimals, for $x = 0(\cdot 0^4 1)\cdot 00999$ to 12 decimals, and for

$$x = 0(\cdot 0^4 1)\cdot 0^3 1(\cdot 0^3 1)\cdot 0^2 1(\cdot 0^2 1)\cdot 01(\cdot 01)\cdot 1(\cdot 1)\cdot 9$$

to 15 decimals. The main table thus covers a range of nearly* $8\pi$ in $x$, at an interval as small as, or smaller than, those used in the best of previous tables; subtraction of multiples of $\pi$ is thus avoided in very many cases by use of these invaluable tables. As with (2), no differences are provided; if needed, a second difference may be found by omission of the last six digits of an entry. An auxiliary table gives second difference coefficients in Bessel's interpolation formula, for every 0·001 of the tabular interval.

Both these volumes (2) and (3) also give conversion tables between nonagesimal and radian measure, but to 6 decimals only.

All three volumes have interesting and informative introductions describing the computation, checking and preparation of the tables; methods of use and interpolation are also described, with numerical examples. Each volume has a foreword, in (1) by H. T. Davis, in (2) by C. E. Van Orstrand, and in (3) by R. C. Archibald. The first book also contains a bibliography, about seventy items, of tables of circular and hyperbolic tangents and cotangents and of closely related functions.

Little need be said about the production of these volumes, from typescript by a photo-offset process, or about their accuracy. These are fully discussed in the review of three other New York M.T.P. volumes which appeared recently †; the remarks made there may be taken to apply to these volumes also. The reproduction has steadily improved, so that (1) is a little better in this respect than (2) or (3). No tests of accuracy have been applied by the writer, but in view of the results of tests previously reported † and of the exhaustive tests applied by the N.Y.M.T.P. during computation, there is every reason to trust the accuracy of the tables. As evidence of scrutiny by the writer, rather than as a criticism of the tables, we note that in (2), p. 223, sinh 1·1115 should have a 1 before the decimal point; nothing more serious has been found.

One criticism, which applies to (1) only, may be made. In the main table, the manner of splitting the digits of an entry into groups is very variable, and might possibly lead to confusion in the use of the tables. Usually the first 5 significant figures are separated from the rest, but this is not done consistently; it seems to the writer safer to split always according to the number of decimals, usually at the fifth.

To sum up, these three New York volumes make a set of tables of outstanding merit, particularly in extent, but also in other respects. No one who has to make much use of radian argument can afford to be without them, and many others also could use them with advantage. J. C. P. MILLER.

* With two more pages (in a total of 250) this could have read "more than."

† *Mathematical Gazette*, 29 (February, 1945), pp. 29–33.

**Exercises in Practical Business Arithmetic.** By H. F. Hemstock. Pp. 96. 2s. 6d. 1944. (Harrap)

In his Preface the author states that: " It is assumed that the students who use this book will be conversant with the basic principles of Arithmetic, and its aim is to develop the student's arithmetical ability in accordance with modern business requirements. . . ." " The exercises are designed primarily for post-secondary school students of School Certificate standard preparing for a business career."

The book is divided into seven sections, the first two, Decimalisation of Money and Cost Calculations each with 350 exercises, and the third on Percentages with 500 exercises of varying difficulty and variety provide excellent preparation for what follows.

In Section 4—" Invoices, Cost Calculations—a more difficult type arranged as invoice items involving percentage and increase on list prices ", we have the introduction of the office " atmosphere " which is extended in the next section as " Business Statistics ". This provides " practice in tabulating statistics of various kinds ", and again, the examples are both numerous and comprehensive.

Section 6—" Practical Assignments "—is the climax of the completion of the task, for here are examples of various types of business records which are usually kept in specially ruled forms—the Petty Cash Book, Sale Analysis, Stock Books and Wages Sheets allowing for computations from Time Sheets, deductions for Insurance and P.A.Y.E., for which tables are given. To close, there is a section " Mental Arithmetic " with 250 examples of a more advanced type of the opening sections.

" Explanatory texts are reduced to a minimum "; nevertheless, the student with a credit in School Certificate arithmetic should follow the course quite well, and having mastered it will be qualified for the work intended.

There is a complete set of Answers, pp. 77–96. These pages are perforated for removal if desired. E. J. A.

## BOOKS RECEIVED FOR REVIEW.

**W. L. Ferrar.** *Higher algebra for schools.* Pp. viii, 214. 12s. 6d. 1945. (Oxford University Press)

**E. F. Freundlich.** *Air navigation.* Pp. vi, 112. 7s. 6d. 1945. (Oliver and Boyd)

**R. W. M. Gibbs.** *Stage A geometry.* 5th edition, with an introduction to navigation by W. M. Symon. Pp. vii, 128. Without answers, 3s.; with answers, 3s. 6d., 1944. (Black)

**S. L. Green.** *An introduction to differential equations.* Pp. 139. 7s. 6d. 1945. (University Tutorial Press)

**G. Hoyland.** *The tyranny of mathematics.* Pp. 52. Paper cover, 1s. 6d. 1945. (Student Christian Movement Press, 56, Bloomsbury St., W.C. 1)

**J. von Neumann and O. Morgenstern.** *Theory of games and economic behaviour.* Pp. xviii, 625. 66s. 6d. 1944. (Princeton University Press; Oxford University Press)

**R. F. Pickard.** *Time, number and the atom.* Pp. vii, 92. 8s. 6d. 1945. (Williams and Norgate)

WANTED: A copy of the *Report on the Teaching of Algebra in Schools* (temporarily out of print); and,

FOR SALE: Bound copy of de la Vallée Poussin, *Cours d'Analyse Infinitésimale*, I, II (1926). C. W. Eaves, 9 Wheatroyd Lane, Almondbury, Huddersfield, Yorks.

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p. 96.

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